Homework I Solution

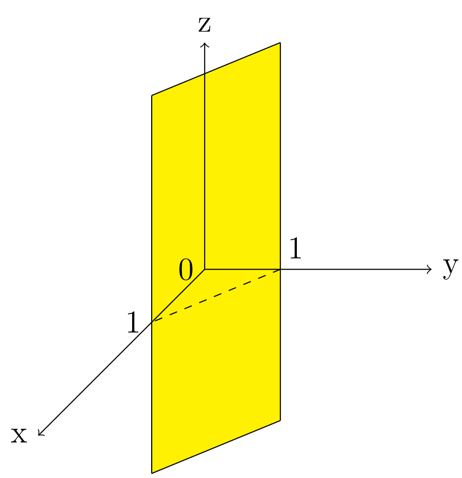
First-Half

1.

$$x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$$
$$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 + 8z + 16 = 15 + 1 + 4 + 16$$
$$(x-1)^2 + (y-2)^2 + (z+4)^2 = 36 = 6^2$$

So this equation represents the sphere centered at $(1, 2, -4)$ with the radius to be 6

2.



3.

$$
\begin{aligned}
2\vec{u} - 3\vec{v} &= 2 < 3, -2, 5 > -3 < -1, 4, 3 > \\
&=< 6, -4, 10 > - < -3, 12, 9 > \\
&=< 9, -16, 1 >
\end{aligned}
$$

4. We can take $-\frac{1}{|\vec{v}|}\vec{v} = -\frac{1}{3} < 2, 2, -1 >=< -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} >$

5.

$$
\begin{aligned}
\lambda\vec{a} + \mu\vec{b} + \eta\vec{c} &= \lambda < 1, 1, -1 > +\mu < 1, -1, 1 > +\eta < -1, 1, 1 > \\
&=< \lambda + \mu - \eta, \lambda - \mu + \eta, -\lambda + \mu + \eta >
\end{aligned}
$$

So if $\vec{v} =< -6, 12, -2 >= \lambda\vec{a} + \mu\vec{b} + \eta\vec{c}$, we get

$$
\begin{cases}
\lambda + \mu - \eta = -6 \\
\lambda - \mu + \eta = 12 \\
-\lambda + \mu + \eta = -2
\end{cases}
$$

Solving the equations, we get $\lambda = 3, \mu = -4, \eta = 5$

6. $< -5, 3, 6 > \cdot < 6, -8, 9 >= (-5) \times 6 + 3 \times (-8) + 6 \times 9 = 0$, so the vectors are orthogonal.

7. Assume $\vec{u} =< x, y, z >$ is a unit vector ($|\vec{u}| = 1$) that makes an angle of $\frac{\pi}{3}$ with $\vec{v}$ and perpendicular to $\vec{k} =< 0, 0, 1 >$, then

$$0 =< x, y, z > \cdot < 0, 0, 1 >= z$$

so $\vec{u} =< x, y, 0 >$. Also,

$$\frac{1}{2} = \cos\frac{\pi}{3} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{x + \sqrt{3}y}{4}$$

We get $x + \sqrt{3}y = 2$. Together with $|\vec{u}| = x^2 + y^2 = 1$, we get $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$, so $\vec{u} =< \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 >$

8. The scalar projection is $\frac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \frac{9}{7}$

   The vector projection is $\frac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\vec{u} = \frac{9}{49} < 3, 6, -2 > = < \frac{27}{49}, \frac{54}{49}, -\frac{18}{49} >$

9. Method I:

   If $\vec{u}+\vec{v} \perp \vec{u}-\vec{v}$, then

$$\begin{aligned}
(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= 0\\
\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{v} &= 0\\
\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} &= 0\\
|\vec{u}|^2 - |\vec{v}|^2 &= 0\\
|\vec{u}|^2 &= |\vec{v}|^2\\
|\vec{u}| &= |\vec{v}|
\end{aligned}$$

   Method II:

   Assume $\vec{u} = < x_1, y_1, z_1 >$, $\vec{v} = < x_2, y_2, z_2 >$.

$$\begin{aligned}
(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= 0\\
< x_1+x_2, y_1+y_2, z_1+z_2 > \cdot < x_1-x_2, y_1-y_2, z_1-z_2 > &= 0\\
(x_1+x_2)(x_1-x_2) + (y_1+y_2)(y_1-y_2) + (z_1+z_2)(z_1-z_2) &= 0\\
x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2 &= 0\\
x_1^2 + y_1^2 + z_1^2 &= x_2^2 + y_2^2 + z_2^2\\
|\vec{u}|^2 &= |\vec{v}|^2\\
|\vec{u}| &= |\vec{v}|
\end{aligned}$$

Second-Half

1. $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -2 & 1 & 3 \end{vmatrix} =< 2, -11, 5 >$

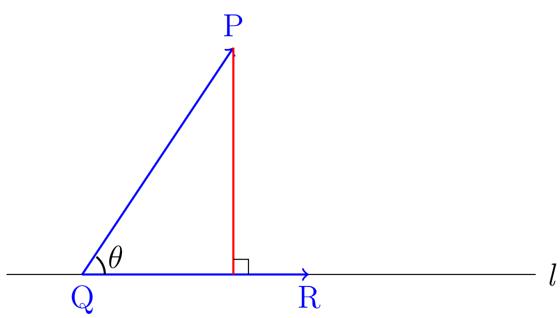
2. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

   $\sqrt{3} = \vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

   $3 = |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

   So the quotient of the above equations implies $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{3}}{3}$, and $0 \leq \theta \leq \pi$, hence $\theta = \frac{\pi}{3}$

3. Let θ be the angle formed by the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$. Then the distance from P to l is $d = |\overrightarrow{QP}| \sin\theta$. So $d = |\overrightarrow{QP}| \sin\theta = \frac{|\overrightarrow{QR} \times \overrightarrow{QP}|}{|\overrightarrow{QR}|}$



4. $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 9$, so the volume is 9

5. $((1-\lambda)1 + \lambda 4, (1-\lambda)2 + \lambda 5, (1-\lambda)3 + \lambda 6)$, $\lambda \in \mathbb{R}$

   i.e. $(1 + 3\lambda, 2 + 3\lambda, 3 + 3\lambda)$, $\lambda \in \mathbb{R}$

6. If the line is perpendicular to the plane, then it will be parallel to the normal vector of the plane, which is $\vec{n} =< 1, -1, 3 >$.

   So the equation of the line is $(2 + \lambda, 4 - \lambda, 6 + 3\lambda)$, $\lambda \in \mathbb{R}$

7. $\frac{x-1}{-1} = \frac{x-5}{2} = \frac{x-6}{3}$

8. $0(x-1)+1(y-2)+4(z-3)=0$, i.e. $y+4z-14=0$

9. Let $P=(0,2,4)$, $Q=(1,-3,2)$, $R=(-3,-2,1)$. Then $\overrightarrow{PQ} =< 1,-5,-2 >$ and $\overrightarrow{PR} =< -3,-4,-3 >$ are parallel to the plane, so $\overrightarrow{PQ} \times \overrightarrow{PR} =< 7,9,-19 >$ is a normal vector to the plane. So the equation of the plane is given by

   $7(x-0)+9(y-2)-19(z-4)=0$, i.e. $7x+9y-19z+58=0$

10. Let $P=(1,2,4)$. We can arbitrarily pick a point on the plane, say $Q=(0,0,5)$. Then $\overrightarrow{PQ} =< -1,-2,1 >$. By the equation of the plane, we see $\vec{n} =< 3,2,1 >$ is a normal vector to the plane.

    So the distance from P to the plane is $\frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|-6|}{\sqrt{14}} = \frac{3}{7}\sqrt{14}$

11. From the equation of the straight line, we see the straight line is parallel to $\vec{v} =< 3,4,12 >$ and $Q=(1,-2,2)$ is on the straight line. Let $P=(1,2,4)$, then $\overrightarrow{PQ} =< 0,-4,-2 >$, and the distance from P to the straight line is

$$\frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|} = \frac{| < 40,-6,12 > |}{13} = \frac{2}{13}\sqrt{445}$$

12. The first plane has normal vector $\vec{n}_1 =< 1,2,3 >$, and the second plane has normal vector $\vec{n}_2 =< -1,2,-3 >$. So the intersection line should be parallel to $\vec{n}_1 \times \vec{n}_2 =< -12,0,4 >$.

    A point in the intersection should satisfy both plane equations

$$\begin{cases} x+2y+3z=1 \\ -x+2y-3z=2 \end{cases}$$

    If we let $z=0$, we get $x=-\frac{1}{2}$ and $y=\frac{3}{4}$. So $(-\frac{1}{2},\frac{3}{4},0)$ is a point in the intersection line.

    So the equation of the intersection is:

    $(-\frac{1}{2}-12\lambda,\frac{3}{4},4\lambda)$, $\lambda \in R$