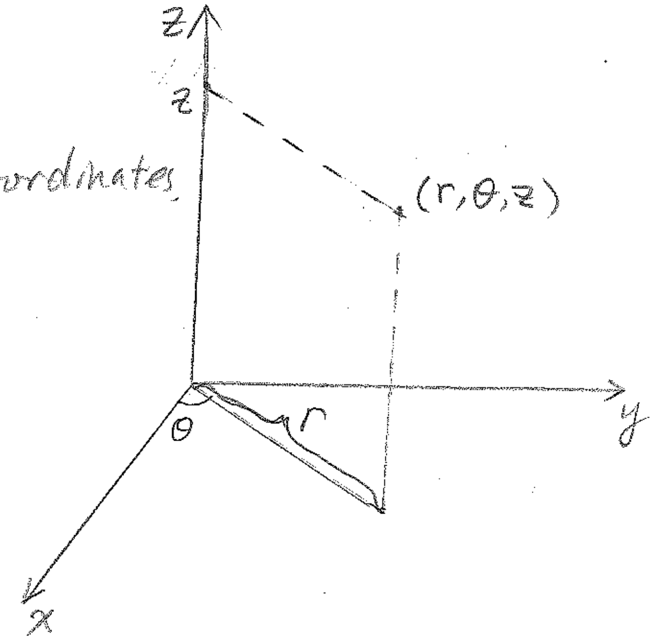
CYLINDRICAL COORDINATES

A point in the three dimensional space can be described in different coordinate systems, which is also the case for two dimensional space. Recall that we have introduced the polar coordinates for $\mathbb{R}^2$, which sometimes provides a more convenient description for some functions and regions, especially when we consider the double integrals.

There are two different ways of generalizing the polar coordinates to three dimensional spaces: Cylindrical Coordinates and Spherical Coordinates

The cylindrical coordinates generalizes the polar coordinates in a "parallel" way, that is, we make a partition of $\mathbb{R}^3$ into horizontal planes, each corresponds to a number z on the z-axis. Then on each horizontal plane, we describe the location of a point by the polar coordinates.

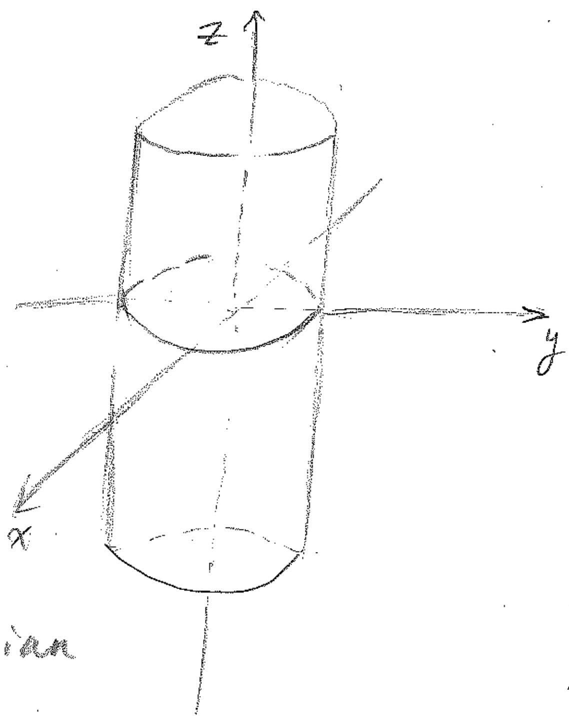


So for a point (x, y, z) in Cartisian Coordinates

$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$

Example. Describe the surface in $\mathbb{R}^3$ whose equation in cylindrical coordinates is $r = 1$

This is the set of points whose projection to the xy-plane has distance 1 to the origin, so each horizontal segment is a circle centered at z-axis and radius 1, it implies the surface is a cylinder whose equation is $x^2+y^2=1$ in Cartisian Coordinates



We can use the cylindrical coordinates to compute triple integrals by applying the change of variable formula.

$\iiint_E f(x,y,z)\,dV$ is an integral, and

if we want to do a change of variable by the cylindrical coordinates, the Jacobian

is $$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial r} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

so we can compute the integral by

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(r\cos\theta, r\sin\theta, z)\, r\, dV'$$

Example. A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point (x,y,z) is proportional to its distance from the z-axis of the cylinder, i.e $K\cdot\sqrt{x^2+y^2}$. (K is a positive constant). Compute the mass of E.

$E = \{(r,\theta,z) \mid 0 \le \theta < 2\pi,\ 0 \le r \le 1,\ 1-r^2 \le z \le 4\}$.

so the mass is $$\iiint_E K\sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 Kr\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} K r^2 \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 K r^2 z \Big|_{z=1-r^2}^{z=4} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 K r^2 (3+r^2)\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 K(3r^2+r^4)\,dr\,d\theta$$

$$= \int_0^{2\pi} K\,d\theta \cdot \int_0^1 3r^2+r^4\,dr$$

$$= \frac{12\pi K}{5}$$

Example. Evaluating $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$

We need to first figure out the region:

It's the cylinder $x^2+y^2=4$ bounded by $z=2$ and the cone $z=\sqrt{x^2+y^2}$.

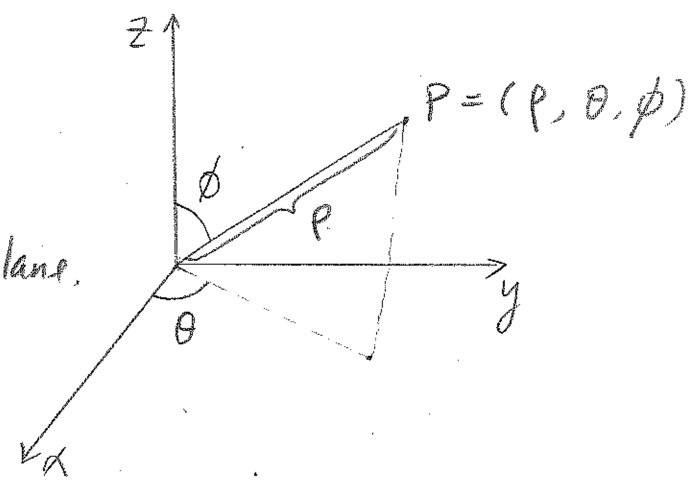
So the region can be written in cylindrical coordinates:

$$E = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r \le z \le 2\}$$

So $\iiint_E (x^2+y^2)\,dz = \int_0^2\int_0^{2\pi}\int_r^2 r^2 \cdot r\,dz\,d\theta\,dr = \frac{16}{5}\pi$

SPHERICAL COORDINATES

We now introduce another generalization of the polar coordinates, which is called the spherical coordinates. In this coordinate, we use three letters (ρ, θ, ϕ) to specify the position of a point in space, as shown in the following picture:



ρ is the length of $\vec{OP}$. θ is the angle of the projection of P on the xy-plane. ϕ is the angle between the positive z-axis and $\vec{OP}$.

The transformation map between the sperical coordinates and the Cartisian Coordinates is given by the following equations:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Also we have $x^2 + y^2 + z^2 = \rho^2$

Example. Use spherical coordinate to represent the following objects:

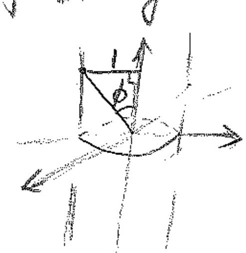
(i) the sphere of radius R, centered at origin

(ii) the region bounded by the cylinder $x^2 + y^2 = 1$

(i). The equation of sphere is $x^2 + y^2 + z^2 = R^2$

so in spherical coordinates, it's $\rho^2 = R^2$. i.e. $\rho = R$

(ii) We see for any non-vertical ray starting from O, with direction specified by θ & ϕ, the length of the segment inside the cylinder is $\rho = \frac{1}{\sin\phi}$ so the region is



$$\{(\rho, \theta, \phi) \mid 0 \le \theta < 2\pi, \ 0 < \phi < \pi, \ 0 \le \rho \le \frac{1}{\sin\phi}\}$$

Now we use spherical coordinates to evaluate Triple Integrals:

$$\frac{\partial x \partial y \partial z}{\partial \rho \partial \theta \partial \phi} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \sin\phi\cos\theta \begin{vmatrix} \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ 0 & -\rho\sin\phi \end{vmatrix} + \rho\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{vmatrix} + \rho\cos\phi\cos\theta \begin{vmatrix} \sin\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & 0 \end{vmatrix}$$

$$= -\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\cos^2\theta$$

$$= -\rho^2\sin\phi(\sin^2\phi + \cos^2\phi)\cos^2\theta - \rho^2\sin\phi\sin^2\theta$$

$$= -\rho^2\sin\phi(\cos^2\theta + \sin^2\theta)$$

$= -\rho^2\sin\phi$, since $\phi \in [0, \pi]$, $\sin\phi \geq 0$. So $\left|\frac{\partial x \partial y \partial z}{\partial \rho \partial \theta \partial \phi}\right| = \rho^2 \sin\phi$

So $\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,dV'$

Example. Compute the volume of a ball of radius R.

We may put the ball's center at Origin, $B = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq R\}$

$$\iiint_B 1\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^R 1 \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\rho$$

$$= \left(\int_0^R \rho^2 d\rho\right)\left(\int_0^{2\pi} 1\, d\theta\right) \cdot \left(\int_0^{\pi} \sin\phi\, d\phi\right)$$

$$= \frac{R^3}{3} \cdot 2\pi \cdot 2$$

$$= \frac{4\pi}{3} R^3$$

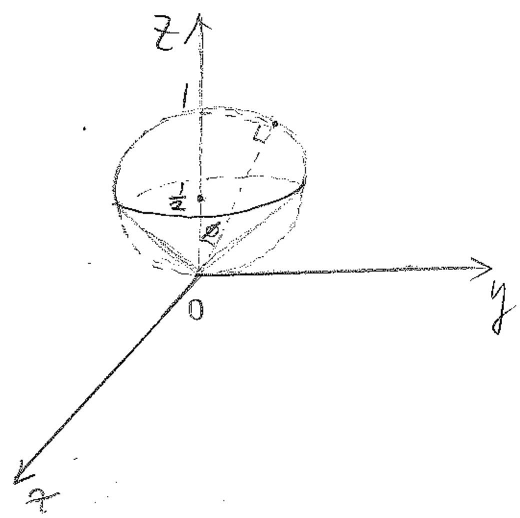
Example. Use spherical coordinates to find the integral

$$\iiint_E x^2+y^2+z^2 \, dV.$$

where E is the region that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$

The sphere is $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$

So it's centered at $(0,0,\frac{1}{2})$ with radius $\frac{1}{2}$.



The region is given by

$$E=\{(r,\theta,\phi) \mid 0\le\phi\le\frac{\pi}{4},\ 0\le\theta<2\pi,\ 0\le\rho\le\cos\phi\}$$

So
$$\iiint_E x^2+y^2+z^2\,dV=\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\cos\phi}\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\cos\phi}\rho^4\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\frac{1}{5}\cos^5\phi\sin\phi\,d\theta\,d\phi$$

$$=\frac{2\pi}{5}\int_0^{\frac{\pi}{4}}\cos^5\phi\sin\phi\,d\phi$$

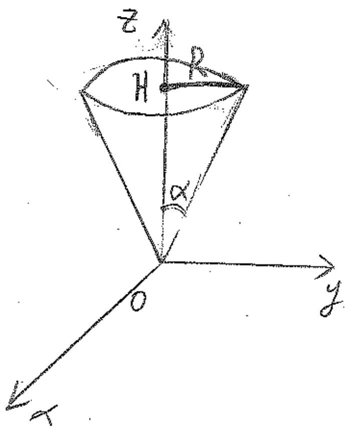
$$=-\frac{2\pi}{5}\int_0^{\frac{\pi}{4}}\cos^5\phi\,d\cos\phi$$

$$=-\frac{2}{5}\pi\cdot\frac{\cos^6\phi}{6}\Big|_0^{\frac{\pi}{4}}$$

$$=-\frac{\pi}{15}\cdot\left(\frac{\sqrt{2}}{2}\right)^6$$

$$=-\frac{\pi}{120}$$

Example. Compute the volume of a cone with height H and radius of the base circle R.



We first try to get a formula for the side surface of the cone:

A point (x,y,z) is on the side surface of the cone if and only if $\frac{\sqrt{x^2+y^2}}{z} = \frac{R}{H} = \tan\alpha$

So the equation is $z = \frac{H}{R}\sqrt{x^2+y^2}$

Then the volume of the cone is

$$\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{\frac{H}{R}\sqrt{x^2+y^2}}^{H} 1 \, dz\,dy\,dx$$

We can use cylindrical coordinates to compute:

$$\iiint_E 1\,dV = \int_0^H\int_0^{2\pi}\int_0^{\frac{z}{H}R} 1\cdot r\,dr\,d\theta\,dz$$

$$= \int_0^H\int_0^{2\pi} \left.\frac{r^2}{2}\right|_{r=0}^{r=\frac{z}{H}R} d\theta\,dz$$

$$= \int_0^H\int_0^{2\pi} \frac{R^2}{2H^2}z^2\,d\theta\,dz$$

$$= \int_0^H \frac{\pi R^2}{H^2}z^2\,dz$$

$$= \frac{1}{3}\cdot\frac{\pi R^2}{H^2}\cdot H^3$$

$$= \frac{\pi R^2 H}{3}$$

We can also use the spherical coordinates:

$$\iiint_E 1\, dV = \int_0^{2\pi} \int_0^{\arctan\frac{R}{H}} \int_0^{\frac{H}{\cos\phi}} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\arctan\frac{R}{H}} \frac{\rho^3}{3} \sin\phi \Big|_{\rho=0}^{\rho=\frac{H}{\cos\phi}} d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\arctan\frac{R}{H}} \frac{H^3}{3} \cdot \frac{1}{\cos^3\phi} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\arctan\frac{R}{H}} \frac{-H^3}{3} \frac{1}{\cos^3\phi} \, d\cos\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{H^3}{3} \cdot \frac{1}{-2} \frac{1}{\cos^2\phi} \Big|_{\phi=0}^{\phi=\arctan\frac{R}{H}} d\theta$$

$$= \int_0^{2\pi} \frac{H^3}{6} \left(\frac{\sqrt{H^2+R^2}}{H}\right)^2 - \frac{H^3}{6} \, d\theta$$

$$= 2\pi \cdot \frac{H^3}{6} \cdot \frac{R^2}{H^2}$$

$$= \frac{\pi R^2 H}{3}$$