CHANGE OF VARIABLES

Given a function $f(x,y)$ and an integral of the function on a region D : $\iint_D f(x,y)\,dA$

We may consider to do a change of variable, that is, to represent x and y as functions of another pair of variables u and v, then translate the integral to be one on the uv-plane.
The integration in polar coordinates gives a particular example of this form. We would like to investigate more general rules for rewriting an integral by a change of variable.

Let T be a transformation from the uv-plane to the xy-plane

$T(u,v) = (x,y)$, where $x = g(u,v)$ and $y = h(u,v)$ both are functions of two variables. (Assume g & h have continuous partial derivatives). We require T to be a one-to-one map, i.e. different points in the uv-plane will be sent to different points on the xy-plane. A good property of a one-to-one transformation is that it has an inverse transformation T^{-1} which satisfies:

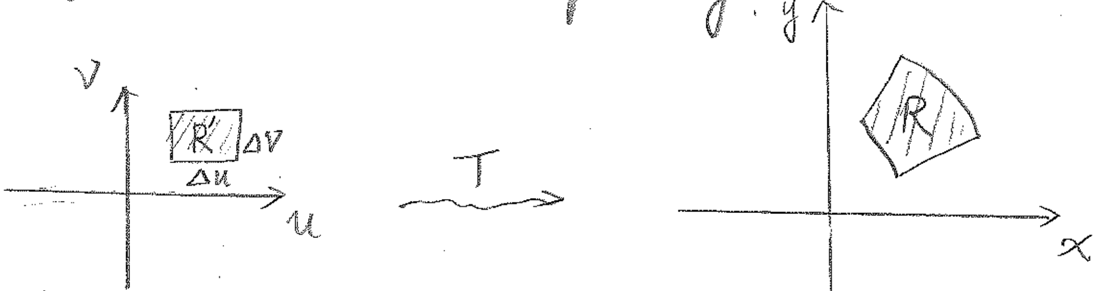
$$(x,y) = T(u,v) \iff (u,v) = T^{-1}(x,y)$$

If a transformation is one-to-one, then we can use it to do a change of variable: given a function $f(x,y)$, consider the integral

$\iint_D f(x,y)\,dA$, we plan to write it as an integral over D', where

$D = T(D')$, D' is a region on the uv-plane.

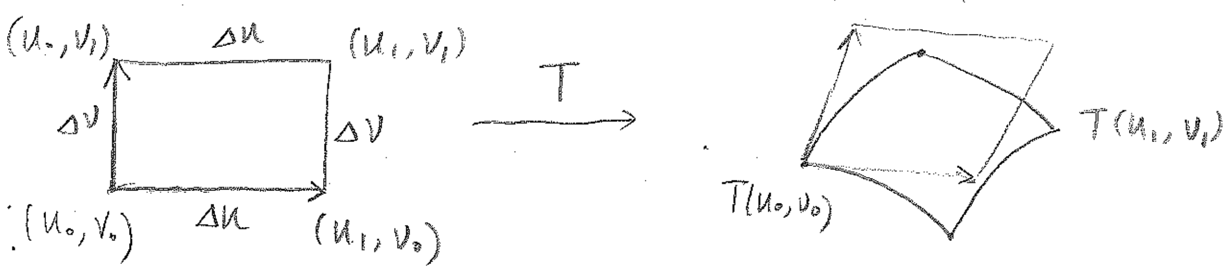
Consider a small rectangular region R' on uv-plane, with length of the edges to be Δu and Δv respectively.



We need to find a way to estimate the area of the image. $R = T(R')$.

Then we can compute $\iint_D f(x,y)\,dA$ by taking the limit of a Riemann Sum over regions of the type R.

The idea to do the estimation is to use the parallelogram formed by the tangent vectors of $T(u,v) = (g(u,v), h(u,v))$



Let $\alpha(u) = T(u, v_0) = (g(u, v_0), h(u, v_0))$

$\beta(v) = T(u_0, v) = (g(u_0, v), h(u_0, v))$

Then tangent vectors at (u_0, v_0) are:

$$\begin{cases} \alpha'(u_0) = \left(\frac{\partial g}{\partial u}(u_0, v_0), \frac{\partial h}{\partial u}(u_0, v_0) \right) \\ \beta'(v_0) = \left(\frac{\partial g}{\partial v}(u_0, v_0), \frac{\partial h}{\partial v}(u_0, v_0) \right) \end{cases}$$

So we use the parallelogram formed by the vectors:

$$\begin{cases} \Delta u\, \alpha'(u_0) := \left\langle \frac{\partial g}{\partial u}(u_0,v_0)\Delta u,\ \frac{\partial h}{\partial u}(u_0,v_0)\Delta u \right\rangle \\ \Delta v\, \beta'(v_0) = \left\langle \frac{\partial g}{\partial v}(u_0,v_0)\Delta v,\ \frac{\partial h}{\partial v}(u_0,v_0)\Delta v \right\rangle \end{cases}$$

to approximate the area R

The area of the parallelogram is given by the cross product:

$$|\Delta u\,\alpha'(u_0)\times\Delta v\,\beta'(v_0)| = \left| \begin{vmatrix} i & j & k \\ \frac{\partial g}{\partial u}(u_0,v_0)\Delta u & \frac{\partial h}{\partial u}(u_0,v_0)\Delta u & 0 \\ \frac{\partial g}{\partial v}(u_0,v_0)\Delta v & \frac{\partial h}{\partial v}(u_0,v_0)\Delta v & 0 \end{vmatrix} \right|$$

$$= \left| \left\langle 0,\ 0,\ \begin{vmatrix} \frac{\partial g}{\partial u}(u_0,v_0)\Delta u & \frac{\partial h}{\partial u}(u_0,v_0)\Delta u \\ \frac{\partial g}{\partial v}(u_0,v_0)\Delta v & \frac{\partial h}{\partial v}(u_0,v_0)\Delta v \end{vmatrix} \right\rangle \right|$$

Define: the Jacobian of the transformation T to be:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$ then

$$|\Delta u\,\alpha'(u_0)\times\Delta v\,\beta'(v_0)| = \left| \frac{\partial(x,y)}{\partial(u,v)}(u_0,v_0) \right| \cdot \Delta u \Delta v .$$

As the cutting is finer, the above expression is closer to that of R.

So the Riemann Sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i^*,v_j^*),\ h(u_i^*,v_j^*)) \left| \frac{\partial(x,y)}{\partial(u,v)}(u_i^*,v_j^*) \right| \Delta u_i \Delta v_j$$

will converge to $\iint_D f(x,y)\,dA$ as $\max \Delta u_i, \Delta v_j \longrightarrow 0$.

On the other hand, this Riemann Sum corresponds to the integral

$$\iint_{R'} f(g(u,v),\ h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

We therefore obtain the following theorem:

Suppose that T is a one-to-one transformation whose Jacobian is nonzero and that maps a region R' in the uv-plane onto a region R in the xy-plane. Then:

$$\iint_R f(x,y)\,dA = \iint_{R'} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA.$$

Example. We use the polar coordinates to test this theorem:

$x = r\cos\theta$, $y = r\sin\theta$. so

$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dA$$

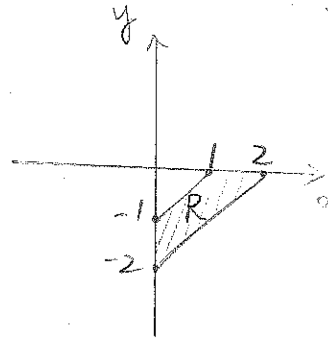
$$= \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\, dA.$$

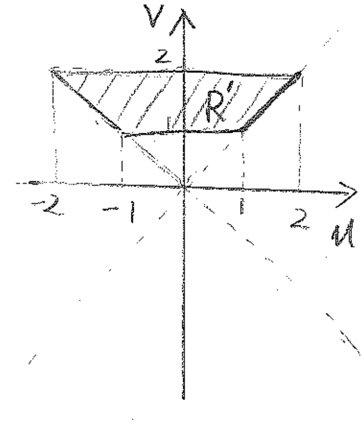
where $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

Example Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\,dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$

We let $u = x+y$ and $v = x-y$.

then $\begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$





$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

so $\left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| = \dfrac{1}{2}$

$$\iint_R f(x,y)\,dA = \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \cdot \frac{1}{2}\,du\,dv = \int_1^2 \frac{v}{2} e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv$$

$$= \int_1^2 \frac{v}{2}(e - e^{-1})\,dv$$

$$= (e - e^{-1}) \cdot \frac{v^2}{4} \Big|_{v=1}^{v=2}$$

$$= \frac{3(e-e^{-1})}{4}$$

There are also generalized version of change of variable for triple integral

If T is a one-to-one transformation that maps a region R' in uvw-space to a region R in xyz-space by

$$(x,y,z) = T(u,v,w) = (g(u,v,w),\ h(u,v,w),\ k(u,v,w)),$$

The Jacobian of T is:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Then:

$$\iiint_R f(x,y,z)\,dV = \iiint_{R'} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

We will use this triple integral version for the discussion of integrals in cylindrical and spherical coordinates.