TRIPLE INTEGRALS

Similar to the case of double integral, we can also integrate a function $f(x,y,z)$ of 3 variables over a 3-dimensional region.

We again start with the case of integration on products of intervals, that is, rectangular box $B = \{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$.

We cut each coordinate into small segments to obtain a partition of B consisting of small rectangular boxes:

$$\begin{cases} a = x_0 < x_1 < \cdots < x_l = b \\ c = y_0 < y_1 < \cdots < y_m = d \\ r \le z_0 < z_1 < \cdots < z_n = s \end{cases}$$

The small box $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ has volume $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$.

We get the triple Riemann Sum $\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*) \Delta V_{ijk}$

Then we take the limit to define the triple integral:

$$\iiint_B f(x,y,z)\, dV = \lim_{\max \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k$$

Similar to the Double Integral case, we also have a Fubini's Theorem for triple integrals:

If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz = \int_r^s \int_a^b \int_c^d f(x,y,z)\, dy\, dx\, dz$$

$$= \int_c^d \int_r^s \int_a^b f(x,y,z)\, dx\, dz\, dy = \int_c^d \int_a^b \int_r^s f(x,y,z)\, dz\, dx\, dy$$

$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\, dy\, dz\, dx = \int_a^b \int_c^d \int_r^s f(x,y,z)\, dz\, dy\, dx$$

Example. Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular region $B = [0,1]\times[-1,2]\times[0,3]$

$$\iiint_B xyz^2\,dV = \int_0^3\int_{-1}^2\int_0^1 xyz^2\,dx\,dy\,dz$$

$$= \int_0^3\int_{-1}^2 \left(\frac{x^2yz^2}{2}\Big|_{x=0}^{x=1}\right)dy\,dz$$

$$= \int_0^3\int_{-1}^2 \frac{yz^2}{2}\,dy\,dz$$

$$= \int_0^3\left(\frac{y^2z^2}{4}\Big|_{y=-1}^{y=2}\right)dz$$

$$= \int_0^3 \frac{3}{4}z^2\,dz$$

$$= \frac{z^3}{4}\Big|_{z=0}^{z=3}$$

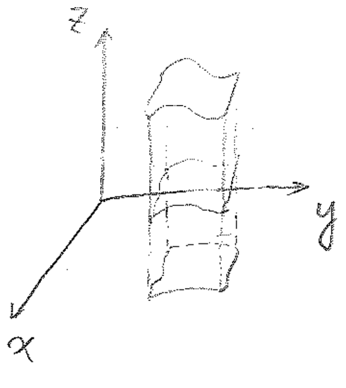
$$= \frac{27}{4}$$

Next, we will define the triple integral over a general bounded region E in $\mathbb{R}^3$. There're several types of regions that we can work with, one of them is given by

$$E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$$

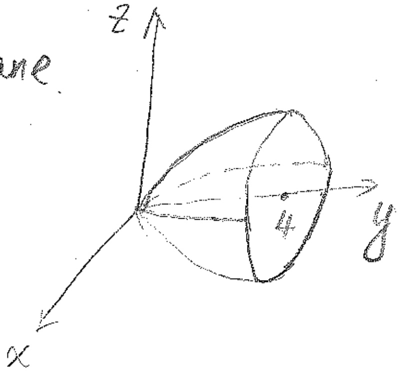
Then $$\iiint_E f(x,y,z)\,dV = \iint_D\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]dA$$

$$= \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$



If we switch the order of dx, dy, dz, there are other types of regions corresponding to them.

Example. Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$.



Consider the projection of the object on x-z plane which is the disk D centered at origin with radius 2. For each (x,z) on D, the solid is bounded by:

$$u_1(x,z) = x^2+z^2 \le y \le u_2(x,z) = 4$$

So the integral is:

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right) dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx$$

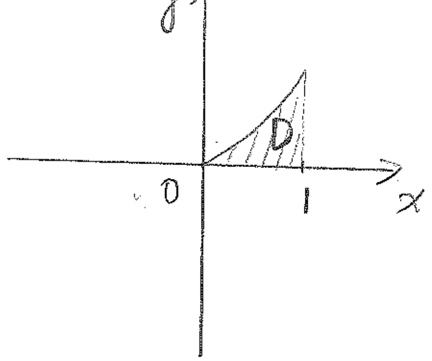
$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+z^2}\,(4-x^2+z^2)\, dz\, dx$$

$$= \int_{0}^{2\pi} \int_{0}^{2} r\,(4-r^2)\cdot r\, dr\, d\theta$$

$$= \frac{128\pi}{15}$$

Example. Express the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

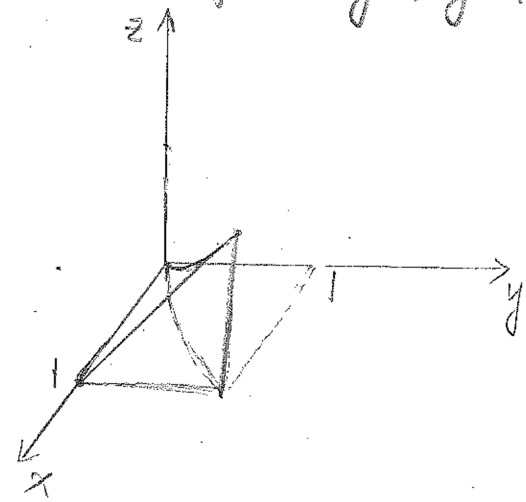
The region E projected onto xy-plane is the 2-dimensional region $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le x^2\}$



Next, for each point in E with projection $(x,y)\in D$, its z coordinate is bounded by $0\le z\le y$ so E is the following region:

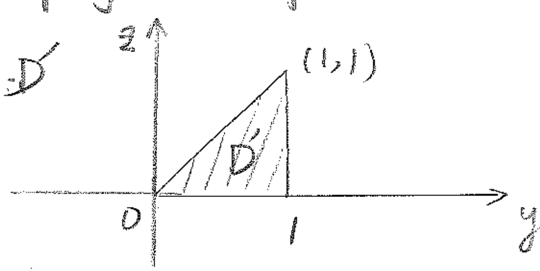
$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx=\iiint_E f(x,y,z)\,dV,$$

where $E=\left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, \begin{array}{l}0\le x\le 1,\ 0\le y\le x^2,\\ 0\le z\le y\end{array}\right\}$



Now we try to write it in a different order $dx\,dz\,dy$

So we need first consider the projection of the solid E onto yz-plane which is the following region D'



and for each point in E whose yz-projection is (x,z), its x-coordinate is bounded by $\sqrt{y}\le x\le 1$.

So the integral is:

$$\iiint_E f(x,y,z)\,dV = \iint_{D'} \left(\int_{\sqrt{y}}^{1} f(x,y,z)\,dx \right) dA = \int_0^1 \int_0^y \int_{\sqrt{y}}^{1} f(x,y,z)\,dx\,dz\,dy$$

A classic application of the triple integral is to compute the mass of an object, given the density function. $\rho(x,y,z)$.

First, how do we interpret density at a point? For a point (x,y,z) inside an object, we consider the part of the object which is the cube centered at (x,y,z) with edge length r (for small r, then entire cube lies inside the object). The density of this cube is $\frac{m(r)}{V(r)}$, where $m(r)$ is the mass of this cube, and $V(r)$ is the volume of this cube. We define the density at (x,y,z) to be

$$\rho(x,y,z) = \lim_{r \to 0} \frac{m(r)}{V(r)}$$

Now if we know the density of an object E, we can approximate its mass by the Riemann Sum:

$$\sum \rho(x^*, y^*, z^*)\,\Delta V$$

and we obtain the mass by taking the limit of the pieces of partition go to zero:

$$\text{Mass} = \iiint_E \rho(x,y,z)\,dV$$

Also, we can compute the volume of a 3-dimensional object by using triple integral: $\mathrm{Vol}(E) = \iiint_E 1\, dV$

Example. Find the volume of a ball of radius r.

We build a coordinate system so that the center of the ball is at origin. Denote the ball by B, then

$$\iiint_B 1\, dV = \iint_D \left(\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} 1\, dz \right) dA$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} 1\, dz\, dy\, dx$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\, dy\, dx \qquad (\text{let } y = \sqrt{r^2-x^2}\sin\theta)$$

$$= \int_{-r}^{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cdot \sqrt{r^2-x^2-(r^2-x^2)\sin^2\theta}\; d\sqrt{r^2-x^2}\sin\theta\, dx$$

$$= \int_{-r}^{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(r^2-x^2)\cos^2\theta\, d\theta\, dx$$

$$= \int_{-r}^{r} (r^2-x^2) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta\, dx \qquad (\text{let } \psi = 2\theta)$$

$$= \int_{-r}^{r} (r^2-x^2) \int_{-\pi}^{\pi} 1+\cos\psi\, d\frac{\psi}{2}\, dx$$

$$= \int_{-r}^{r} \frac{r^2-x^2}{2} (\psi + \sin\psi)\Big|_{-\pi}^{\pi}\, dx$$

$$= \pi \int_{-r}^{r} r^2-x^2\, dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right) = \pi \cdot \frac{4}{3} r^3 = \frac{4}{3}\pi r^3$$