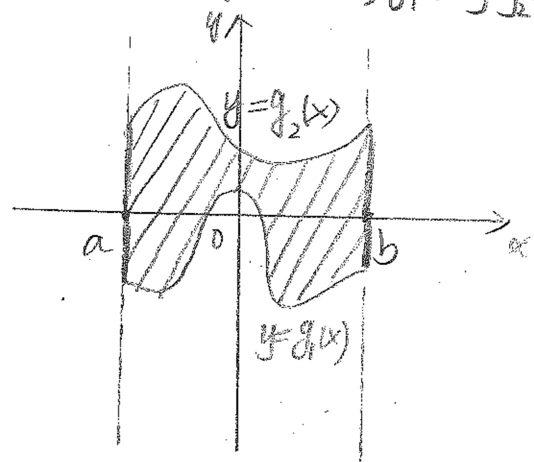
# DOUBLE INTEGRAL OVER GENERAL REGIONS

In many cases, we want to compute the volume of some solid whose base is not a rectangle. In such cases, we need to define and compute the double integral over a general region.

A plane region $D$ is said to be of type I if it lies between the graphs of two continuous functions of $x$, that is $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$




We define $\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$, where

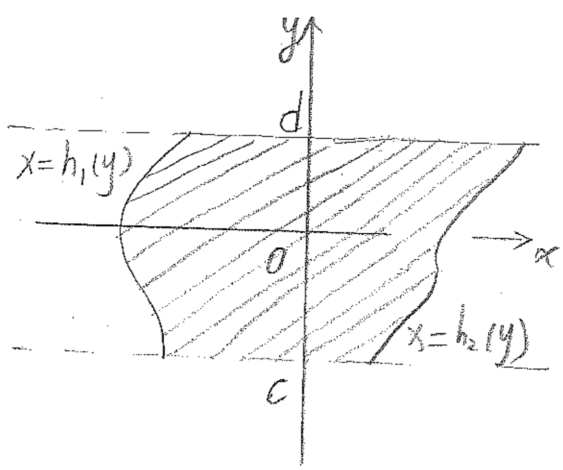
$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \in R \setminus D \end{cases}$$

where $R = [a,b] \times [c,d]$ is a rectangular region including $D$.

By Fubini's Theorem:

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

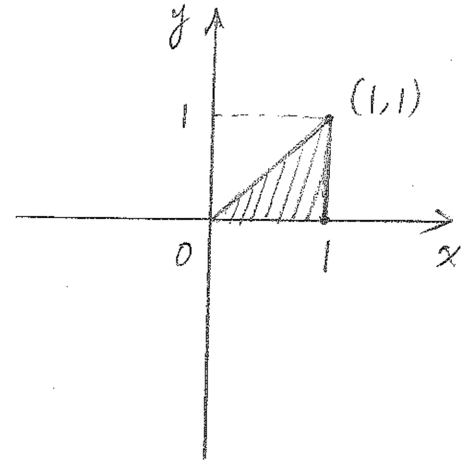
Similarly, we can define a plane region to be of type II if it lies between the graphs of two continuous functions of $y$, that is
$D = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$




and the integral is given by

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example. Compute the integral $\iint_D f(x,y)\,dA$ over the triangular region with vertices $(0,0)$, $(1,0)$, $(1,1)$, where $f(x,y) = xy$.




If we use type I:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^x xy\,dy\,dx$$

$$= \int_0^1 \left( \frac{x}{2} y^2 \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_0^1 \left( \frac{x^3}{2} \right) dx$$

$$= \frac{1}{8} x^4 \Big|_{x=0}^{x=1}$$

$$= \frac{1}{8}$$

If we use type II:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_y^1 xy\,dx\,dy$$

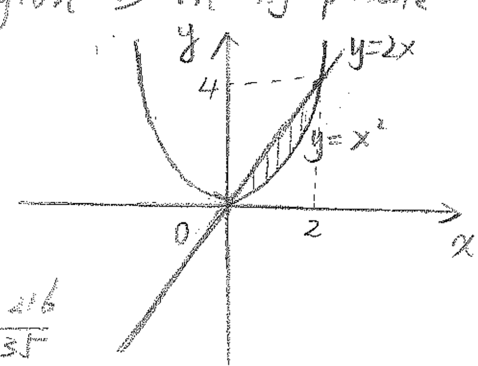
$$= \int_0^1 \frac{y}{2} x^2 \Big|_{x=y}^{x=1} dy$$

$$= \int_0^1 \left( \frac{y}{2} - \frac{y^3}{2} \right) dy$$

$$= \frac{y^2}{4} - \frac{y^4}{8} \Big|_{y=0}^{y=1}$$

$$= \frac{1}{8}$$

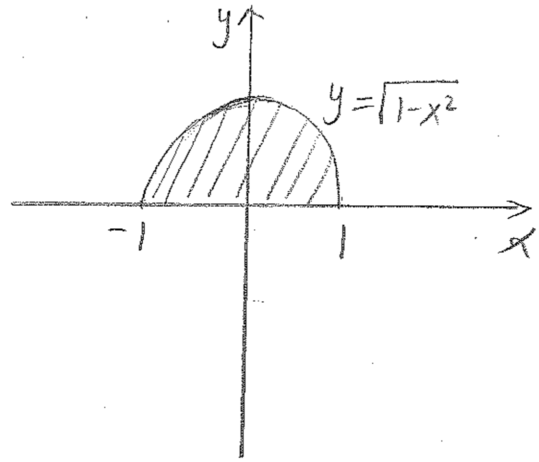
Example. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region $D$ in $xy$-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.




$$\iint_D f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} x^2 + y^2\,dy\,dx$$

$$= \int_0^2 \left( -\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Big|_0^2 = \frac{216}{35}$$

**Example** Rewrite the type I integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$ as a type II integral.




We see the region $D$ is enclosed by the x-axis and the upper half of the unit circle.

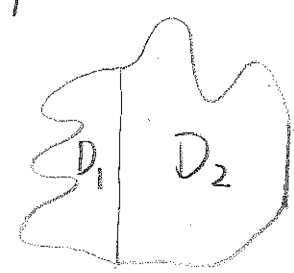
So the type II integral is

$$\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

**Properties of Double Integrals.**

- $\iint_D f(x,y)+g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$
- $\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA$
- If $f(x,y) \geq g(x,y)$ for all $(x,y)\in D$, then $\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$
- If $D = D_1 \cup D_2$ and $D_1, D_2$ do not overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$




- $\iint_D 1\,dA = A(D)$
- If $m \leq f(x,y) \leq M$ for all $(x,y)\in D$, then

$$mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$$

# DOUBLE INTEGRALS IN POLAR COORDINATES

Recall that apart from the cartisian $xy$-coordinate, the points on a plane can also be described by the polar coordinates. $(r, \theta)$

The conversion between them is given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \begin{cases} 0, & \text{if } (x,y) \text{ is on positive } x\text{-axis} \\ \arctan\frac{y}{x}, & \text{if } (x,y) \text{ is inside Quadrant I.} \\ \frac{\pi}{2}, & \text{if } (x,y) \text{ is on positive } y\text{-axis} \\ \pi + \arctan\frac{y}{x}, & \text{if } (x,y) \text{ is inside Quadrant II or Quadrant III, or on negative } x\text{-axis} \\ \frac{3}{2}\pi, & \text{if } (x,y) \text{ is on negative } y\text{-axis} \\ 2\pi + \arctan\frac{y}{x}, & \text{if } (x,y) \text{ is inside Quadrant IV.} \end{cases} \end{cases}$$

Sometimes, regions are easier to describe in polar coordinates than in Cartisian coordinates.

For example: the region between the circle of radius 1 and the circle of radius 2, both centered at origin, is described by $1 \le r \le 2$. in polar coordinates.

We would like to investigate how to do integration if the region is described by polar coordinates

The method turns out to be the following theorem, which we'll discuss about:

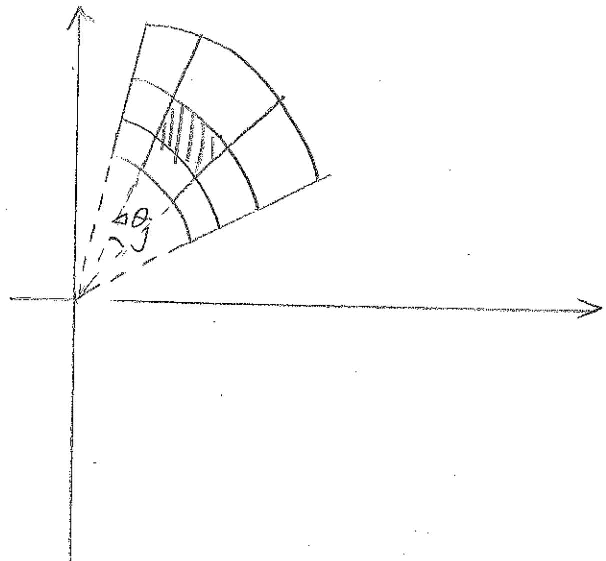
Theorem. If $f(x,y)$ is continuous on a polar rectangle $R$ given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

First, we will call the following type of set a polar rectangle:

$$R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$

Using idea similar to Riemann Sum, we divide $R$ into $m \times n$ sub-polar rectangles by dividing $[a, b]$ into $m$ subintervals $[r_{i-1}, r_i]$ and $[\alpha, \beta]$ into $n$ subintervals $[\theta_{i-1}, \theta_i]$, and denote $\Delta r_i = r_i - r_{i-1}$, $\Delta\theta_j = \theta_j - \theta_{j-1}$. Now we denote the region corresponding to $[r_{i-1}, r_i]$ & $[\theta_{j-1}, \theta_j]$ to be $R_{ij} = \{(r, \theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1}\ \theta \le \theta_j\}$.




The area of $R_{ij}$, denoted by $\Delta A_{ij}$, can be computed as:

$$\Delta A_{ij} = \frac{r_i^2}{2}\Delta\theta_j - \frac{r_{i-1}^2}{2}\Delta\theta_j$$

$$= \frac{\Delta\theta_j}{2}(r_i - r_{i-1})(r_i + r_{i+1})$$

Define $r_i^* = \dfrac{r_{i-1} + r_i}{2}$, $\theta_j^* = \dfrac{\theta_{j-1} + \theta_j}{2}$.

then $\Delta A_{ij} = r_i^* \Delta r_i \Delta\theta_j$

So a Riemann Sum for the polar coordinates is given by:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^* \Delta r_i \Delta\theta_j$$

This implies:

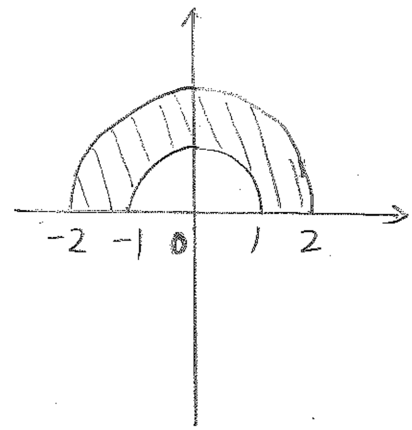
$$\iint_R f(x,y)\,dA = \lim_{\substack{\max\Delta r_i \to 0 \\ \max\Delta\theta_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^* \Delta r_i \Delta\theta_j$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

So we proved the theorem.

Example. Evaluate $\iint_R (3x+4y^2)\,dA$, where $R$ is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

The region $R=\{(r,\theta) \mid 1\le r\le 2,\ 0\le\theta\le\pi\}$.




$$\text{so } \iint_R (3x+4y^2) = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$

$$= \int_0^\pi (r^3\cos\theta + r^4\sin^2\theta)\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\sin^2\theta\, d\theta$$

$$= 7\int_0^\pi \cos\theta\, d\theta + 15\int_0^\pi \sin^2\theta\, d\theta$$

$$= 7\sin\theta\Big|_0^\pi + 15\cdot\frac{1}{4}\sin 2\theta\Big|_0^\pi$$

$$= \frac{15}{2}\pi$$

Similar to the integration in cartisian coordinates, the integration on polar coordinates can also be extended to more complicated regions:

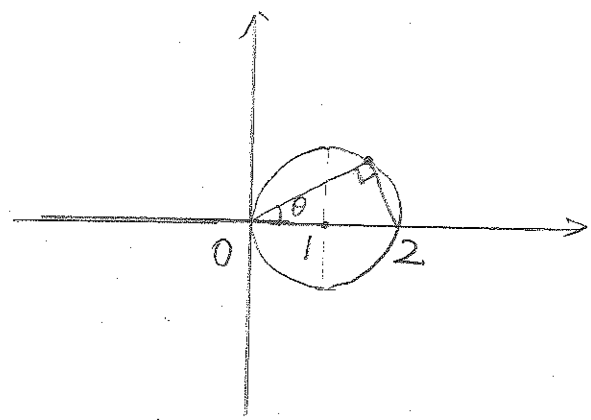
Theorem. If $f$ is continuous on a polar region of the form

$$D=\{(r,\theta) \mid \alpha\le\theta\le\beta,\ h_1(\theta)\le r\le h_2(\theta)\},$$

then $$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

Example. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above $xy$-plane, and inside the cylinder $x^2+y^2=2x$

The cylinder is $(x-1)^2+y^2=1$, so the solid is the part above the disk $(x-1)^2+y^2\le 1$ on $xy$-plane and below the graph of $z=x^2+y^2$




the disk can be represented by

$D=\{(r,\theta)\mid -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le r\le 2\cos\theta\}$

so the volume of the solid is

$$\iint_D x^2+y^2\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\cdot r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4}\,d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta \qquad (\text{Note } \cos^4\theta \text{ is an even function})$$

$$= 8\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1+2\cos 2\theta+\cos^2 2\theta\,d2\theta \qquad (\text{let } \psi=2\theta)$$

$$= \int_0^{\pi} 1+2\cos\psi+\cos^2\psi\,d\psi$$

$$= \psi+2\sin\psi+\frac{1}{2}\left(\psi+\frac{1}{2}\sin 2\psi\right)\Big|_{\psi=0}^{\psi=\pi} = \frac{3}{2}\pi$$