DOUBLE INTEGRALS OVER RECTANGLES

$f(x,y)$ is a two-variable function defined on a closed rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b], y \in [c,d]\}$.

Suppose $f(x,y) \geq 0$, then the graph of $f(x,y)$ is above the xy plane. Consider the solid that lies above R and under the graph of f. We would like to compute its volume.

The strategy is to cut R into many smaller rectangular regions, and approximate the volume of each region by a cuboid. then sum them up. When the cutting is finer and finer, the sum of the volume of these small cuboid is closer and closer to the volume of the original region.

Now: we divide $[a,b]$ into $a = x_0 < x_1 < \cdots < x_m < b$.

divide $[c,d]$ into $c = y_0 < y_1 < \cdots < y_n < d$.

then the region R is divided into mn regions

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \text{ with area } \Delta A_{ij} = \Delta x_i \Delta y_j$$

For each R_{ij}, we choose a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$, then the solid above R_{ij} and below the graph of $f(x,y)$ can be approximated by

$f(x_i^*, y_j^*) \cdot \Delta x_i \Delta y_j$.

and if we sum up all these pieces. the volume is approximated by

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta x_i \Delta y_j.$$

Now taking the limit as $\max \Delta x_i \to 0$, $\max \Delta y_j \to 0$

we expect the volume is $\lim\limits_{\max \Delta x_i, \Delta y_j \to 0} f(x_i^*, y_j^*) \Delta x_i \Delta y_j$.

If the limit exists, we define it to be the double integral of f over the rectangular region R, and denote it by

$$\iint_R f(x,y)\,dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A_{ij}$$

If we define a more general volume, that is, if $f(x,y)$ is a two variable function (may not be positive-valued), we define the signed volume of the solid bounded between the graph of the function and the x-y plane to be the volume of the part above the xy-plane minus the volume of the part below the xy-plane.

So for any two variable function $f(x,y)$, we can consider the limit $\lim_{\substack{\max \Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A_{ij}$, if the limit exists, we define it to be the double integral of f over the rectangular region R, and denote it by $\iint_R f(x,y)\,dA$.

A function $f(x,y)$ is called integrable if such limit exists.

Theorem: If $f(x,y)$ is bounded and continuous except on a finite number of curves on the region, then $f(x,y)$ is integrable on this region.

So many of the two variable functions that we are familiar with are integrable on rectangular regions.

Since $\iint_R f(x,y)\,dA$ is defined as the limit of $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij}$.

We can choose some m & n, as well as x_i^* and y_j^* to approximate the value of $\iint_R f(x,y)\,dA$. One choice for x_i^*, y_j^* is to take the midpoint on each small rectangle:

Midpoint Rule for Double Integrals:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A, \quad \text{where } \begin{cases} \bar{x}_i \text{ is the midpoint of } [x_{i-1}, x_i] \\ \bar{y}_j \text{ is the midpoint of } [y_{j-1}, y_j] \end{cases}$$

Example. Use Midpoint Rule with $m=n=2$ to estimate $\iint_R (x-3y^2)\,dA$, where $R=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ 1\le y\le 2\}$

$$\begin{cases} x_0=0,\ x_1=1,\ x_2=2 \\ y_0=1,\ y_1=\frac{3}{2},\ y_2=2 \end{cases} \Rightarrow \begin{cases} \bar{x}_1=\frac{1}{2}, & \bar{x}_2=\frac{3}{2} \\ \bar{y}_1=\frac{5}{4}, & \bar{y}_2=\frac{7}{4} \end{cases}$$

so $\Delta A_{ij} = 1\times\frac{1}{2}=\frac{1}{2}$

$$\iint_R (x-3y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\,dA_{ij}$$

$$= f(\bar{x}_1,\bar{y}_1)\cdot\frac{1}{2} + f(\bar{x}_1,\bar{y}_2)\cdot\frac{1}{2} + f(\bar{x}_2,\bar{y}_1)\frac{1}{2} + f(\bar{x}_2,\bar{y}_2)\cdot\frac{1}{2}$$

$$= \frac{1}{2}\left[f(\tfrac{1}{2},\tfrac{5}{4}) + f(\tfrac{1}{2},\tfrac{7}{4}) + f(\tfrac{3}{2},\tfrac{5}{4}) + f(\tfrac{3}{2},\tfrac{7}{4})\right]$$

$$= -11.875$$

By approximations, we can have an idea about how large is the value of an integral. but sometimes we need the exact number rather than an approximation, so we would like to find a way which is more convenient for computation.

Iterated Integrals.

$f(x,y)$ is a function on two variables and continuous on $R=[a,b]\times[c,d]$, define $\int_c^d f(x,y)\,dy$ to be the integration with respect to y, viewing x as a constant. so the result should be a function of x, i.e. $A(x)=\int_c^d f(x,y)\,dy$ Then we integrate $A(x)$ with respect to x.

$$\int_a^b A(x)\,dx=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

This form of integral on the right side of the above equation is called an iterated integral. This is an important construction because of the following theorem:

Fubini's Theorem:
If f is a continuous function on a rectangle $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy.$$

More generally, this is true when $f(x,y)$ is bounded on R, and discontinuous only on a finite number of smooth curves.

The proof of the above theorem is very deep in mathematics, but its intuition is clear: (This is not a rigorous proof!)

$$\iint_R f(x,y)\,dA=\lim_{\substack{\max\Delta x_i\to 0\\ \Delta y_j\to 0}}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_j^*)\,dA_{ij}$$

$$=\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{m}\left(\lim_{\max\Delta y_j\to 0}\sum_{j=1}^{n} f(x_i^*,y_j^*)\,\Delta y_j\right)\Delta x_i$$

$$=\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{m}\left(\int_c^d f(x_i^*,y)\,dy\right)\Delta x_i$$

$$=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

Example. Evaluate $\iint_R y\sin(xy)\,dA$, where $R=[1,2]\times[0,\pi]$

$$\iint_R y\sin(xy)\,dA=\int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$=\int_0^{\pi}\left(-\cos(xy)\Big|_{x=1}^{x=2}\right)dy$$

$$=\int_0^{\pi}(-\cos 2y+\cos y)\,dy$$

$$=-\frac{1}{2}\sin 2y+\sin y\Big|_{y=0}^{y=\pi}$$

$$=0$$

Example. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the plane $x=2$, the plane $y=2$, and the three coordinate planes.

S is the solid lies under the graph of $z=f(x,y)=16-x^2-2y^2$ and above the square $R=[0,2]\times[0,2]$

so $V=\iint_R f(x,y)\,dA=\int_0^2\int_0^2(16-x^2-2y^2)\,dx\,dy$

$$=\int_0^2 (16-2y^2)x-\frac{1}{3}x^3\Big|_{x=0}^{x=2}dy$$

$$=\int_0^2 32-4y^2-\frac{8}{3}\,dy$$

$$=\int_0^2\left(\frac{88}{3}-4y^2\right)dy$$

$$=\frac{88}{3}y-\frac{4}{3}y^3\Big|_{y=0}^{y=2}$$

$$=48$$

Properties of double integrals:

- If $f(x,y)=g(x)h(y)$, then $\iint\limits_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\cdot\left(\int_c^d h(y)\,dy\right)$

- $\iint\limits_R f(x,y)+g(x,y)\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA$

- $\iint\limits_R c f(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$

- $f(x,y)\geq g(x,y)$ for all $(x,y)\in R$, then $\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA$.

Example. If $R=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$, compute $\iint\limits_R \sin x\cos y\,dA$.

$$\iint\limits_R \sin x\cos y\,dA = \int_0^{\frac{\pi}{2}} \sin x\,dx \cdot \int_0^{\frac{\pi}{2}} \cos x\,dx = 1$$