EXTREME VALUES

A function of two variables has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b) (i.e. there exists a disk on the x-y plane centered at (a,b) such that if (x,y) is in the disk then $f(x,y) \le f(a,b)$). We call $f(a,b)$ a local maximum value.

Similarly, we can define the local minimum at (a,b) if $f(x,y) \ge f(a,b)$ when (x,y) in near (a,b).

A necessary condition for the existence of local extreme is the following:

Theorem. If f has a local maximum or minimum at (a,b) and the first order partial derivatives of f exist there, then

$$\frac{\partial f}{\partial x}(a,b)=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b)=0$$

The idea for the proof is that if (a,b) is a local maximum/minimum of f, then restricted to the lines $x=a$ and $y=b$, it's still a locall maximum/minimum.

We define (a,b) a critical point of f if $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$, or at least one of the partial derivatives doesn't exist.

So if (a,b) is a local maximum/minimum, it must be a critical point. But the other way is not true in general: a critical point may not be a local maximum/minimum.

After we obtain all the critical points, there is a way to tell if they're local extreme or not, by the following theorem:

Second Derivative Test:

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $\frac{\partial f}{\partial x}(a,b)=0$, $\frac{\partial f}{\partial y}(a,b)=0$.

Let $D = D(a,b) = \frac{\partial^2 f}{\partial x^2}(a,b)\frac{\partial^2 f}{\partial y^2}(a,b) - \left[\frac{\partial^2 f}{\partial x \partial y}(a,b)\right]^2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$

Then:

(i) if $D>0$, $\frac{\partial^2 f}{\partial x^2}(a,b)>0$, then $f(a,b)$ is a local minimum

(ii) if $D>0$, $\frac{\partial^2 f}{\partial x^2}(a,b)<0$, then $f(a,b)$ is a local maximum.

(iii) if $D<0$, then $f(a,b)$ is a saddle point.

In the above theorem, a saddle point is a critical point which is not a local extreme.

Example. $f(x,y)=y^2-x^2$

$\frac{\partial f}{\partial x} = -2x$ $\frac{\partial f}{\partial y} = 2y$. So $(0,0)$ is the only critical point.

$\frac{\partial^2 f}{\partial x^2} = -2$ $\frac{\partial^2 f}{\partial y^2} = 2$ $\frac{\partial^2 f}{\partial x \partial y} = 0$.

So $D(0,0) = (-2)\times 2 - 0^2 = -4 < 0$, $(0,0)$ is a saddle point.

we conclude there's no local extreme, hence no extreme.

Example. Find the shortest distance from $(1,0,-2)$ to the plane $x+2y+z=4$.

Points on the plane are of the form $(x, y, 4-x-2y)$.

So the distance between $(1,0,-2)$ and $(x,y,4-x-2y)$ is

$$d = \sqrt{(1-x)^2 + (0-y)^2 + (-2-(4-x-2y))^2}$$

Instead of minimizing d, we can minimize $d^2 = (x-1)^2 + y^2 + (x+2y-6)^2$

So let $f(x,y) = (x-1)^2 + y^2 + (x+2y-6)^2$

Let $\begin{cases} \frac{\partial f}{\partial x}(x,y) = 4x+4y-14=0 \\ \frac{\partial f}{\partial y}(x,y) = 4x+10y-24=0 \end{cases} \Rightarrow \begin{cases} x = \frac{11}{6} \\ y = \frac{5}{3} \end{cases}$

so the only critical point is $(\frac{11}{6}, \frac{5}{3})$.

$\frac{\partial^2 f}{\partial x^2} = 4$, $\frac{\partial^2 f}{\partial y^2} = 10$. $\frac{\partial^2 f}{\partial x \partial y} = 4$. so $D(x,y) = 4\times 10 - 4^2 = 24 > 0$.

$(\frac{11}{6}, \frac{5}{3})$ is a local minimum point.

Since this is the only local minimum, and we kow the function has minimum, so the function must obtains minimum at this minimum point.

so the shortest distance is $d = \sqrt{f(\frac{11}{6}, \frac{5}{3})} = \frac{5}{6}\sqrt{6}$

Maximum & Minimum on a closed and bounded set:

We say a set in $\mathbb{R}^2$ is closed if it contains all the boundary points.
We say a set in $\mathbb{R}^2$ is bounded if it's contained in some disk.

Theorem. If $f(x,y)$ is continuous on a closed and bounded set S in $\mathbb{R}^2$, then f attains an absolute maximum at some point (x_1, y_1) and absolute minimum at some point (x_2, y_2) on S.

So the maximum and minimum always exist on a closed and bounded set; the question is how to find them. A point can be either an interior point or a boundary point, so we use the following strategy:

① Find the critical points and values on the interior of S.

② Find the maximum & minimum values on the boundary of S

③ Compare all the extreme values obtained in ① & ②.

Example. Find the absolute maximum and minimum values of $f(x,y)=x^2-2xy+2y$ on the rectangle $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 3,\ 0\le y\le 2\}$

① Critical points: $\begin{cases} \frac{\partial f}{\partial x} = 2x-2y=0 \\ \frac{\partial f}{\partial y} = -2x+2=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$

So the only critical point is $(1,1)$, $f(1,1)=1$.

② On the line segment $(0,0)$ — $(3,0)$:

$f(x,0)=x^2$ so min: $f(0,0)=0$. max: $f(3,0)=9$

On the line segment $(3,0)$ — $(3,2)$:

$f(3,y)=9-4y$ so min: $f(3,2)=1$, max: $f(3,0)=9$

On the line segment $(0,2)$ — $(3,2)$:

$f(x,2)=x^2-4x+4 = (x-2)^2$. So min: $f(2,2)=0$. max: $f(0,2)=4$

On the line segment $(0,0)$ — $(0,2)$:

$f(0,y)=2y$. so min: $f(0,0)=0$, max: $f(0,2)=4$

③ By ①, ②. we see f obtains maximum value 9 at $(3,0)$ and minimum value 0 at $(0,0)$ and $(2,2)$.

OPTIMIZATION WITH CONSTRAINT

Sometimes we need to maximize or minimize a function under certain constrains. For example, we want to maximize the area of a rectangle given that the perimeter is fixed. One way to solve this type of problem is to express one variable in terms of the others, and reduce the problem into one with one less variables. But in general, it may not be easy to express one variable explicitly as a formula of the other variables. So we need to develop a more general method.

Method of Lagrange Multipliers:

To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$ (assuming that extreme values exist, $\nabla g \neq 0$ on $g(x,y,z)=k$):

(i) Solve for $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z)=k \end{cases}$

(We call λ the Lagrange Multiplier)

(ii) The maximum point and minimum point are among the solutions in (i).

Example. A rectangular box without lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.

Let the length, width and height of the box be x, y, z

then $V = xyz$ is the volume.

The Constraint is $g(x,y,z) = 2xz + 2yz + xy = 12$

So $\nabla V = \langle yz, xz, xy \rangle$ $\nabla g = \langle 2z+y, 2z+x, 2x+2y \rangle$

$$\begin{cases} \nabla V = \lambda \nabla g \\ 2xz + 2yz + xy = 12 \end{cases}$$

Which becomes
$$\begin{cases} yz = \lambda(2z+y) \\ xz = \lambda(2z+x) \\ xy = \lambda(2x+2y) \\ 2xz+2yz+xy=12 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 2 \\ z = 1 \\ \lambda = \frac{1}{2} \end{cases}$$

So we conclude the maximum is obtained at $x=2$, $y=2$, $z=1$.
the volume is 4.

Two Constraints:

When there are two constraints $g(x,y,z)=k$ and $h(x,y,z)=c$ for the function $f(x,y,z)$, in order to find maximum and minimum. (assume ∇g and ∇h are non-zero and not parallel):

(i). Solve the system of equations
$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = k \\ h(x,y,z) = c \end{cases}$$

(ii) The maximum and minimum are among the solutions in (i)

Example. Find the maximum value of $f(x,y,z) = x+2y+3z$ on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$

Let $g(x,y,z) = x-y+z=1$. $h(x,y,z) = x^2+y^2=1$.

$\nabla f = \langle 1,2,3\rangle$ $\nabla g = \langle 1,-1,1\rangle$ $\nabla h = \langle 2x, 2y, 0\rangle$

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ x-y+z=1 \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} 1 = \lambda + 2\mu x \\ 2 = -\lambda + 2\mu y \\ 3 = \lambda \\ x-y+z=1 \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} x = -\frac{2}{\sqrt{29}} \\ y = \frac{5}{\sqrt{29}} \\ z = 1+\frac{7}{\sqrt{29}} \\ \lambda = 3 \\ \mu = \frac{\sqrt{29}}{2} \end{cases} \text{ or } \begin{cases} x = \frac{2}{\sqrt{29}} \\ y = -\frac{5}{\sqrt{29}} \\ z = 1-\frac{7}{\sqrt{29}} \\ \lambda = 3 \\ \mu = -\frac{\sqrt{29}}{2} \end{cases}$$

$\Downarrow$ $f(x,y,z) = 3+\sqrt{29}$ $\Downarrow$ $f(x,y,z) = 3-\sqrt{29}$

So the maximum value is $3+\sqrt{29}$