DIRECTIONAL DERIVATIVES

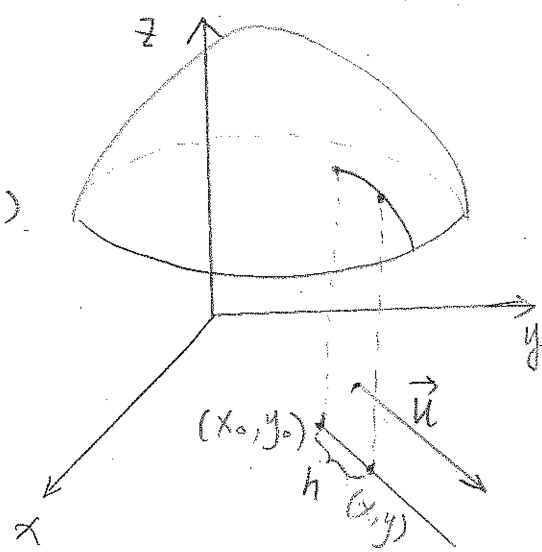
The partial derivatives of a multi-variable function, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, tell us the rate of change of the function along x-direction and y-direction.

But what about other directions?

On the x-y plane, each direction can be represented by a unit vector $\vec{u}$.

We are going to define the directional derivative of a function $z=f(x,y)$ at (x_0, y_0) in the direction $\vec{u}$:

On the x-y plane, consider the line l passing through (x_0, y_0) and parallel to the unit vector $\vec{u}$. Passing through the line l, there is a unique vertical plane α, and α intersects the graph of $z=f(x,y)$ along a curve C, so C projects to l on the x-y plane.



If we start at (x_0, y_0) and travel along $\vec{u}$ direction for a distance h, and arrive at (x, y).
Then the vector with initial point (x_0, y_0) and terminal point (x,y) is $\langle x-x_0, y-y_0\rangle = h\vec{u}$

If we know $\vec{u} = \langle a, b\rangle$ $(a^2+b^2=1)$, then

$\langle x-x_0, y-y_0\rangle = h\vec{u} = \langle ha, hb\rangle$

$$\Rightarrow \begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases} \Rightarrow f(x,y) = f(x_0+ha, y_0+hb)$$

So the rate of change of the function along $\vec{u}$ direction at (x_0, y_0) is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$$

We call it the directional direvative of f at (x_0, y_0) in the direction $\vec{u}$

There is a faster way to compute the directional derivative, if we know the following theorem:

Theorem. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x}(x,y)\cdot a + \frac{\partial f}{\partial y}(x,y)\cdot b$$

Proof. For a fixed (x_0, y_0) in the domain of f, we define the function

$$g(h) = f(x_0 + ha, y_0 + hb)$$

Then we get $g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$

$$= D_{\vec{u}}(x_0, y_0)$$

But we can also write $g(h) = f(x,y)$. $x = x_0 + ha$, $y = y_0 + hb$

So the Chain Rule implies $g'(h) = \frac{\partial f}{\partial x}(x,y)\cdot\frac{dx}{dh} + \frac{\partial f}{\partial y}(x,y)\cdot\frac{dy}{dh}$

$$= a\frac{\partial f}{\partial x}(x,y) + b\frac{\partial f}{\partial y}(x,y)$$

When $h=0$, $x = x_0$, $y = y_0$. So $g'(0) = a\frac{\partial f}{\partial x}(x_0, y_0) + b\cdot\frac{\partial f}{\partial y}(x_0, y_0)$

If the unit vector $\vec{u}$ forms an angle θ with the positive x-axis, then $\vec{u} = \langle \cos\theta, \sin\theta \rangle$. So we can compute the directional direvative along $\vec{u}$ by $D_{\vec{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\cdot\cos\theta + \frac{\partial f}{\partial y}(x_0,y_0)\sin\theta$.

Example. Find the directional derivative $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$, and $\vec{u}$ is the unit vector given by angle $\theta = \frac{\pi}{6}$. What is $D_{\vec{u}} f(1,2)$?

$$D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x}(x,y)\cos\frac{\pi}{6} + \frac{\partial f}{\partial y}(x,y)\sin\frac{\pi}{6} = (3x^2 - 3y)\cdot\frac{\sqrt{3}}{2} + (-3x + 8y)\cdot\frac{1}{2}$$

$$D_{\vec{u}} f(1,2) = (3\times 1^2 - 3\times 2)\cdot\frac{\sqrt{3}}{2} + (-3\times 1 + 8\times 2)\cdot\frac{1}{2} = \frac{13 - 3\sqrt{3}}{2}$$

We have seen that $D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x}(x,y) \cdot a + \frac{\partial f}{\partial y}(x,y) \cdot b$ if $\vec{u} = \langle a, b \rangle$ is a unit vector. We can rewrite the above equation as

$$D_{\vec{u}} f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle \cdot \langle a, b \rangle$$

We now define the gradient of f to be the vector

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$

then $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

Example. Find the gradient of the function $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$, and find the directional direvative in the direction of the vector $\vec{v} = \langle 2, 5 \rangle$.

$\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 - 4 \rangle$ so $\nabla f(2,-1) = \langle -4, 8 \rangle$

Note $\vec{v}$ is not a unit vector, so we first compute the unit vector in the direction of $\vec{v}$, which is $\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

So $D_{\vec{u}} f(2,-1) = \nabla f(2,-1) \cdot \vec{v} = \langle -4, 8 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle = \frac{32}{\sqrt{29}}$

We can extend the above constructions of directional derivatives and gradient vectors to functions of n variables (in particular, 3 variables)

The gradient gives a solution to the maximization of directional derivatives:

Theorem. If f is a differentiable function of n variables, then the maximum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $|\nabla f(x_1, \ldots, x_n)|$, and it's achieved when $\vec{u} = \frac{\nabla f(x_1, \ldots, x_n)}{|\nabla f(x_1, \ldots, x_n)|}$

Proof. $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cdot \cos\theta = |\nabla f| \cdot \cos\theta$

So $D_{\vec{u}} f$ obtains maximum when $\theta = 0$.

Example. $f(x,y) = xe^y$. In what direction does f have the maximum rate of change at $(2,0)$? What's the maximum rate of change?

$\nabla f(x,y) = \langle e^y, xe^y \rangle$, so $\nabla f(2,0) = \langle 1, 2 \rangle$

The maximum rate of change is along $\nabla f(2,0) = \langle 1, 2 \rangle$ direction

The maximum rate of change is $|\nabla f(2,0)| = \sqrt{5}$.

The gradient has another important application, which is related to level sets:

Theorem. If $F(x_1, \dots, x_n) = C$ is a level set then ∇f is a normal vector for this level set in the space $\mathbb{R}^n$. In particular, when $n=2$, $\nabla f(x,y)$ is perpendicular to the tangent line of $f(x,y) = C$ at (x,y); when $n=3$, $\nabla f(x,y,z)$ is perpendicular to the tangent plane of $f(x,y,z) = C$ at (x,y,z)

Example. Find the equation of the tangent plane at $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$

Let $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$, then $\nabla F(x,y,z) = \langle \frac{x}{2}, 2y, \frac{2}{9}z \rangle$

so $\nabla F(-2,1,-3) = \langle -1, 2, -\frac{2}{3} \rangle$

So the tangent plane has equation

$$-1 \cdot (x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$$

$$\Rightarrow 3x - 6y + 2z + 18 = 0$$