THE CHAIN RULE

Consider a function \( z = F(x, y) \). If \( x \) and \( y \) are both functions of \( t \), say \( x = f(t) \) and \( y = g(t) \), then \( z \) is also a function of \( t \):

\[
z = F(f(t), g(t))
\]

So we can talk about the derivative \( \frac{dz}{dt} \), which can be obtained by the Chain Rule:

**Theorem.** When \( z = f(x, y) \) with \( x = f(t), y = g(t) \), then:

\[
\frac{dz}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}
\]

**Example.** Find \( \frac{dz}{dt} \) when \( z = F(x, y) = x^2 + y^3 \) with \( x = t^2, y = 2t \):

\[
\frac{dz}{dt} = \frac{\partial F}{\partial x} (x^2 + y^3) \cdot \frac{d}{dt} (t^2) + \frac{\partial F}{\partial y} (x^2 + y^3) \cdot \frac{d}{dt} (2t)
\]

\[
= 2x \cdot 2t + 3y^2 \cdot 2
\]

\[
= 2t^2 \cdot 2t + 3(2t)^3 \cdot 2
\]

\[
= 4t^3 + 24t^2
\]

We see this gives the same result if we first write \( z \) in terms of \( t \), then take derivative:

\[
z = x^2 + y^3 = (t^2)^2 + (2t)^3 = t^4 + 8t^3
\]

\[
\frac{dz}{dt} = 4t^3 + 24t^2.
\]

Now let's try to prove the Chain Rule:

\[
\frac{dz}{dt} = \lim_{h \to 0} \frac{F(f(t+h), g(t+h)) - F(f(t), g(t))}{h} + E_2 (g(t+h) - g(t))
\]

\[
= \lim_{h \to 0} \frac{\frac{\partial F}{\partial x} (f(t), g(t)) \cdot (f(t+h) - f(t)) + \frac{\partial F}{\partial y} (f(t), g(t)) \cdot (g(t+h) - g(t)) + E_1 (f(t+h) - f(t))}{h}
\]
\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial y}(x) \cdot \frac{\partial y}{\partial x} + \frac{\partial f}{\partial y}(x) \cdot \frac{\partial y}{\partial y} + \varepsilon_1 \frac{f(x+h) - f(x)}{h} + \varepsilon_2 \frac{\partial f}{\partial y}(x) \cdot \frac{\partial y}{\partial x}
\]

\[
\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \frac{\partial g}{\partial x}(x) + \frac{\partial g}{\partial y}(x) \cdot \frac{\partial y}{\partial x} + \frac{\partial g}{\partial y}(x) \cdot \frac{\partial y}{\partial y} + \varepsilon_1 \frac{g(x+h) - g(x)}{h} + \varepsilon_2 \frac{\partial g}{\partial y}(x) \cdot \frac{\partial y}{\partial x}
\]

\[
\frac{\partial f}{\partial x} = \frac{df}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} + 0 \cdot \frac{df}{dx} + 0 \cdot \frac{df}{dy}
\]

\[
\frac{\partial g}{\partial x} = \frac{dg}{dt} + \frac{dg}{dy} \cdot \frac{dy}{dt}
\]

Example. If \( z = x^2y + 3xy^4 \), where \( x = \sin 2t \) and \( y = \cos 2t \), find \( \frac{dz}{dt}(0) \).

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (2xy + 3y^4) \cdot 2\cos 2t + (x^2 + 12xy^3) \cdot (-2 \sin 2t)
\]

\[
= 2(2xy + 3y^4) \cos 2t - 2(x^2 + 12xy^3) \sin 2t.
\]

When \( t = 0 \), \( x = \sin(2 \cdot 0) = 0 \), \( y = \cos(2 \cdot 0) = 1 \), so

\[
\frac{dz}{dt}(1) = 2(2 \cdot 0 \cdot 1 + 3 \cdot 1^4) \cos(2 \cdot 0) - 2(0^2 + 12 \cdot 0 \cdot 1^3) \sin(2 \cdot 0)
\]

\[
= 6
\]

There is another kind of Chain Rule, which deals with the following situation:

Theorem. Suppose \( z = F(x, y) \) is a differentiable function of \( x \) & \( y \), where \( x = f(s, t) \), \( y = g(s, t) \), are differentiable, then:

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}
\]

Example. If \( z = e^{x \cdot \sin y} \), where \( x = st^3 \), \( y = st^4 \). Find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = (e^{x \cdot \sin y}) \cdot (t^2) + (e^{x \cdot \sin y}) \cdot (2st)
\]

\[
= t^2 e^{st^2} \sin (st^3) + 2st e^{st^2} \cos (st^3)
\]

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = (e^{x \cdot \sin y}) \cdot (2st) + (e^{x \cdot \sin y}) \cdot s^4
\]

\[
= 2st e^{st^2} \sin (st^3) + s^4 e^{st^2} \cos (st^3)
\]
In general, we can extend the Chain Rule to the following case:

Theorem. Suppose \( u \) is a differentiable function of the \( n \) variables \( x_1, x_2, \ldots, x_n \) and each \( x_j \) is a differentiable function of the \( m \) variables \( t_1, t_2, \ldots, t_m \). Then \( u \) is a function of \( t_1, t_2, \ldots, t_m \) and

\[
\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}
\]

for each \( i = 1, 2, \ldots, m \).

Example. If \( g(s,t) = f(s^2-t^2, t^2-s^2) \) and \( f \) is differentiable, show that \( g \) satisfies the equation

\[
t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0.
\]

Let \( x = s^2-t^2 \), \( y = t^2-s^2 \), then we see

\( g(s,t) = f(x,y) \) with \( x = s^2-t^2 \), \( y = t^2-s^2 \).

So the chain rule implies:

\[
\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} (2t) + \frac{\partial f}{\partial y} (-2t)
\]

and

\[
\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} (2s) + \frac{\partial f}{\partial y} (-2s)
\]

Then

\[
t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = t \cdot 2s \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) + s \cdot 2t \left( \frac{\partial f}{\partial y} - \frac{\partial f}{\partial x} \right) = 0.
\]
Let $F(x, y)$ be a function of two variables, and consider the level curve $F(x, y) = C$, where $C \in \mathbb{R}$ is a constant. Suppose $y$ is a function of $x$ on part of the level curve, say $y = f(x)$, which is therefore implicitly defined by $F(x, f(x)) = C$. We would like to find $y' = f'(x)$.

Take the derivative with respect to $x$ on both sides of $F(x, y) = C$, we get

$$\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

i.e.

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \quad (\frac{\partial F}{\partial y} \neq 0)$$

so we obtain $f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$.

Example: Find the slope and equation for the tangent line at $(x, y) = (2, 1)$ to the curve $x^3 + x^2y - 2y^3 - 10y = 10$.

Let $F(x, y) = x^3 + x^2y - 2y^3 - 10y$, then we are considering the level set $F(x, y) = 0$.

$$\frac{\partial F}{\partial x} = 3x^2 + 2xy, \quad \frac{\partial F}{\partial y} = x^2 - 4y^2 - 10$$

So

$$\frac{\partial F}{\partial x} (2, 1) = 3\times 2^2 + 2\times 2 \times 1 = 16, \quad \frac{\partial F}{\partial y} (2, 1) = 2^2 - 4\times 1 - 10 = -10$$

$$y' = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{16}{-10} = \frac{8}{5}$$

the slope is $\frac{8}{5}$, the tangent line is $y - 1 = \frac{8}{5}(x - 2)$.
Implicit Differentiation with One More Variable:

Consider a function $F(x, y, z)$ of three variables. For a constant $c \in \mathbb{R}$, the level set $\{ (x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = c \}$ usually forms a surface. Suppose that $z$ is a function of $x$ and $y$ on this surface, say $z = f(x, y)$. Then we can use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$F(x, y, f(x, y)) = c \quad \text{(assume } \frac{\partial F}{\partial z} \neq 0)$$

by the Chain Rule.

$$\begin{align*}
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} &= 0 \\
\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} &= 0
\end{align*} \quad \Rightarrow \quad \begin{cases}
\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \\
\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}
\end{cases}$$

Example. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z$ is implicitly defined as a function of $x$ and $y$ via:

$$x^2 + y^2 + z^2 = 1$$

Let $F(x, y, z) = x^2 + y^2 + z^2$, then $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y$, $\frac{\partial F}{\partial z} = 2z$,

$$\begin{align*}
\frac{\partial z}{\partial x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x}{2z} = -\frac{x}{z} \\
\frac{\partial z}{\partial y} &= -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y}{2z} = -\frac{y}{z}
\end{align*}$$