THE CHAIN RULE

Consider a function $z = F(x,y)$. If x and y are both functions of t, say $x = f(t)$ and $y = g(t)$, then z is also a function of t:

$$z = F(f(t), g(t))$$

So we can talk about the derivative $\frac{dz}{dt}$, which can be obtained by the Chain Rule:

Theorem. When $z = f(x,y)$ with $x = f(t)$, $y = g(t)$, then:

$$\frac{dz}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}$$

Example. Find $\frac{dz}{dt}$ when $z = F(x,y) = x^2 + y^3$ with $x = t^2$, $y = 2t$

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(x^2+y^3) \cdot \frac{d}{dt}(t^2) + \frac{\partial}{\partial y}(x^2+y^3) \cdot \frac{d}{dt}(2t)$$

$$= 2x \cdot 2t + 3y^2 \cdot 2$$

$$= 2 \cdot t^2 \cdot 2t + 3 \cdot (2t)^2 \cdot 2$$

$$= 4t^3 + 24t^2$$

We see this gives the same result if we first write z in terms of t, then take derivative:

$$z = x^2 + y^3 = (t^2)^2 + (2t)^3 = t^4 + 8t^3$$

$$\frac{dz}{dt} = 4t^3 + 24t^2.$$

Now let's try to prove the Chain Rule:

$$\frac{dz}{dt} = \lim_{h \to 0} \frac{F(f(t+h), g(t+h)) - F(f(t), g(t))}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\partial F}{\partial x}(f(t), g(t)) \cdot (f(t+h) - f(t)) + \frac{\partial F}{\partial y}(f(t), g(t))(g(t+h) - g(t)) + \varepsilon_1 (f(t+h) - f(t)) + \varepsilon_2 (g(t+h) - g(t))}{h}$$

$$= \lim_{h\to 0}\left[\frac{\partial F}{\partial x}(f(t),g(t))\cdot\frac{f(t+h)-f(t)}{h} + \frac{\partial F}{\partial y}(f(t),g(t))\cdot\frac{g(t+h)-g(t)}{h} + \varepsilon_1\frac{f(t+h)-f(t)}{h} + \varepsilon_2\frac{g(t+h)-g(t)}{h}\right]$$

$$= \frac{\partial F}{\partial x}\cdot\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} + \frac{\partial F}{\partial y}\cdot\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} + \lim_{h\to 0}\varepsilon_1\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} + \lim_{h\to 0}\varepsilon_2\lim_{h\to 0}\frac{g(t+h)-g(t)}{h}$$

$$= \frac{\partial F}{\partial x}\cdot\frac{df}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dg}{dt} + 0\cdot\frac{df}{dt} + 0\cdot\frac{dg}{dt}$$

$$= \frac{\partial F}{\partial x}\cdot\frac{df}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dg}{dt}$$

Example. If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos 2t$, find $\frac{dz}{dt}(0)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = (2xy+3y^4)\cdot 2\cos 2t + (x^2+12xy^3)\cdot(-2\sin 2t)$$

$$= 2(2xy+3y^4)\cos 2t - 2(x^2+12xy^3)\cdot\sin 2t.$$

when $t=0$. $x = \sin(2\cdot 0) = 0$, $y = \cos(2\cdot 0) = 1$. So

$$\frac{dz}{dt}(1) = 2(2\times 0\times 1 + 3\times 1^4)\cos(2\times 0) - 2(0^2 + 12\times 0\times 1^3)\sin(2\times 0)$$

$$= 6$$

There is another kind of Chain Rule, which deals with the following situation:

Theorem. Suppose $z = F(x,y)$ is a differentiable function of x & y, where $x = f(s,t)$, $y = g(s,t)$, are differentiable, then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Example. If $z = e^x\sin y$, where $x = st^2$, $y = s^2t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (e^x\sin y)\cdot(t^2) + (e^x\cos y)\cdot(2st)$$

$$= t^2e^{st^2}\sin(s^2t) + 2st\,e^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = (e^x\sin y)\cdot(2st) + (e^x\cos y)\cdot s^2$$

$$= 2st\,e^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)$$

In general, we can extend the Chain Rule to the following case:

Theorem. Suppose u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

for each $i = 1, 2, \ldots, m$

Example. If $g(s,t) = f(s^2-t^2, t^2-s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Let. $x = s^2-t^2$. $y = t^2-s^2$. then we see

$$g(s,t) = f(x,y) \text{ with } x = s^2-t^2, \; y = t^2-s^2$$

So the chain rule implies:

$$\begin{cases} \dfrac{\partial g}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} = \dfrac{\partial f}{\partial x}\cdot(-2t) + \dfrac{\partial f}{\partial y}\cdot(2t) \\[2ex] \dfrac{\partial g}{\partial s} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial s} = \dfrac{\partial f}{\partial x}(2s) + \dfrac{\partial f}{\partial y}(-2s) \end{cases}$$

Then $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t\cdot 2s\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) + s\cdot 2t\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right) = 0$

IMPLICIT DIFFERENTIATION

Let $F(x,y)$ be a function of two variables, and consider the level curve $F(x,y)=C$, where $C \in \mathbb{R}$ is a constant. Suppose y is a function of x on part of the level curve, say $y=f(x)$, which is therefore implicitly defined by $F(x,f(x))=C$. We would like to find $y'=f'(x)$.

Take the derivative with respect to x on both sides of $F(x,y)=C$.

we get
$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

i.e.
$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \qquad \left(\frac{\partial F}{\partial y}\neq 0\right)$$

so we obtain $f'(x)=-\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$.

Example: Find the slope and equation for the tangent line at $(x,y)=(2,1)$ to the curve $x^3+x^2y-2y^2-10y=10$

Let $F(x,y)=x^3+x^2y-2y^2-10y$, then we are considering the level set $F(x,y)=0$.

$$\frac{\partial F}{\partial x}=3x^2+2xy, \quad \frac{\partial F}{\partial y}=x^2-4y-10$$

so $\frac{\partial F}{\partial x}(2,1)=3\times 2^2+2\times 2\times 1=16$, $\frac{\partial F}{\partial y}(2,1)=2^2-4\times 1-10=-10$

$$y'=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{16}{-10}=\frac{8}{5}$$

the slope is $\frac{8}{5}$, the tangent line is $y-1=\frac{8}{5}(x-2)$

Implicit Differentiation with One More Variable:

Consider a function $F(x,y,z)$ of three variables. For a constant $c \in \mathbb{R}$, the level set $\{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = c\}$ usually forms a surface. Suppose that z is a function of x and y on this surface, say $z = f(x,y)$. Then we can use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$F(x, y, f(x,y)) = c \qquad \left(\text{assume } \frac{\partial F}{\partial z} \neq 0\right)$$

by the Chain Rule.

$$\begin{cases} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial z}{\partial x} = 0 \\[2ex] \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial z}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \\[2ex] \dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \end{cases}$$

Example. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is implicitly defined as a function of x and y via:

$$x^2 + y^2 + z^2 = 1$$

Let $F(x,y,z) = x^2+y^2+z^2$, then $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y$, $\frac{\partial F}{\partial z} = 2z$,

$$\begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\dfrac{2x}{2z} = -\dfrac{x}{z} \\[2ex] \dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\dfrac{2y}{2z} = -\dfrac{y}{z} \end{cases}$$