PARTIAL DERIVATIVES

Recall that for a single-variable function, $f(x)$, it's derivative measures the rate of change of $y=f(x)$ at x.

For a multi-variable function, we can have an analogue of that, but we need to specify the rate of change is with respect to which variable, since we have more than one variables

If $z=f(x,y)$ is a function with 2 variables we define the partial derivatives

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

In other words, $\frac{\partial f}{\partial x}$ is to take derivative of f w.r.t. x, considering y as a constant; $\frac{\partial f}{\partial y}$ is to take derivative of f w.r.t. y, considering x as a constant.

Example. $f(x,y)=e^{xy}\ln(x+y)$.

$$\frac{\partial f}{\partial x}=ye^{xy}\ln(x+y)+e^{xy}\cdot\frac{1}{x+y}=e^{x+y}\left(y\ln(x+y)+\frac{1}{x+y}\right)$$

$$\frac{\partial f}{\partial y}=xe^{xy}\ln(x+y)+e^{xy}\cdot\frac{1}{x+y}=e^{xy}\left(x\ln(x+y)+\frac{1}{x+y}\right)$$

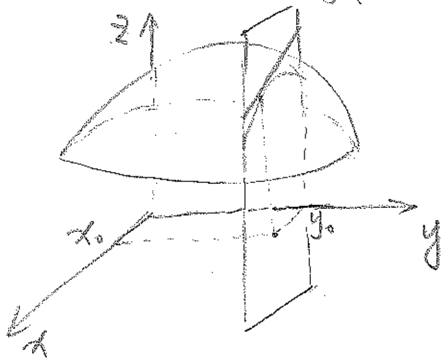
Example. $f(x,y)=\sin\left(\frac{x}{1+y}\right)$.

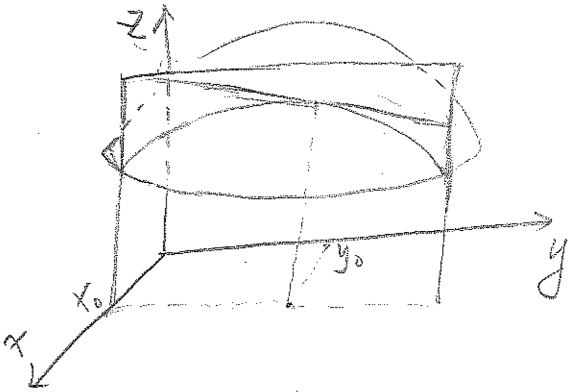
$$\frac{\partial f}{\partial x}=\left(\cos\frac{x}{1+y}\right)\cdot\frac{1}{1+y} \qquad \frac{\partial f}{\partial y}=\left(\cos\frac{x}{1+y}\right)\cdot\left(-\frac{x}{(1+y)^2}\right)=-\left(\cos\frac{x}{1+y}\right)\cdot\frac{x}{(1+y)^2}$$

We can interpret partial derivatives geometrically as the slope of certain curve in the following way:

$z=f(x,y)$ has a graph in the xyz-coordinate system, which consists of points of form $(x,y,f(x,y))$.

At (x_0, y_0), consider the vertical plane $y = y_0$, which intersects the graph of $z = f(x,y)$ at a curve, then the slope of the curve at $x = x_0$ in the plane $y = y_0$ is $\frac{\partial f}{\partial x}(x_0, y_0)$.





Similarly, consider the vertical plane $x = x_0$, which intersects the graph of $z = f(x,y)$ at a curve, then the slope of the curve at $y = y_0$ in the plane $x = x_0$ is $\frac{\partial f}{\partial y}(x_0, y_0)$

Higher Order Derivatives:

If we take the partial derivative of a partial derivative we will get second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Example. $f(x,y) = x \sin y$

$$\frac{\partial f}{\partial x} = \sin y, \quad \frac{\partial f}{\partial y} = x \cos y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\sin y) = 0.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\sin y) = \cos y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x \cos y) = \cos y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x \cos y) = -x \sin x$$

Observe in the previous example, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. This is not a coincidence, but instead, it's implied by the following theorem:

Clairaut's Theorem: f is defined on a disk that contains the point (a,b), If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}$$

For higher order partial derivatives, we can define in a similar way:

$$\frac{\partial^3 f}{\partial x \partial y \partial x}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right), \quad \frac{\partial^4 f}{\partial x \partial y^2 \partial x}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\right)$$

For functions with more than two variables, we can also define partial derivatives and higher order partial derivatives in a same fashion:

For example, if $f(x,y,z)=x^2y+y^2z+z^2x$ then:

$$\frac{\partial f}{\partial x}=2xy+z^2 \quad \frac{\partial f}{\partial y}=x^2+2yz \quad \frac{\partial f}{\partial z}=y^2+2zx$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=2y, \quad \frac{\partial^2 f}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=2x, \quad \frac{\partial^2 f}{\partial x \partial z}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)=2z$$

$$\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=2x, \quad \frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=2z, \quad \frac{\partial^2 f}{\partial y \partial z}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)=2y$$

$$\frac{\partial^2 f}{\partial z \partial x}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)=2z, \quad \frac{\partial^2 f}{\partial z \partial y}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)=2y, \quad \frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)=2x$$

The Clairaut's Theorem can also be extended to more variables and higher order partial derivatives:

$f(x_1, x_2, \dots, x_n)$ is defined on a n-dimensional ball centered at $(a_1, a_2, \dots, a_n)$, If all the n-th order partial derivatives are continuous, then we can switch the order of different partial derivatives for an n-th partial derivative.

TANGENT PLANES & LINEAR APPROXIMATIONS

Recall that when we interpret partial derivatives, we consider the intersection of the graph of $z=f(x,y)$ with $y=y_0$ to obtain a curve, whose tangent line at $x=x_0$ has slope $\frac{\partial f}{\partial x}(x_0,y_0)$. Similarly there's another curve whose tangent line at $y=y_0$ has slope $\frac{\partial f}{\partial y}(x_0,y_0)$.

Now we define the tangent plane of f at (x_0,y_0) to be the plane passing through (x_0,y_0) containing the above two tangent lines.

We know the equation of this plane should be of the form

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \quad , \text{ where } z_0=f(x_0,y_0)$$

where $\langle a,b,c\rangle$ is a normal vector of the plane.

Observe that $\langle 1,0,\frac{\partial f}{\partial x}(x_0,y_0)\rangle$ and $\langle 0,1,\frac{\partial f}{\partial y}(x_0,y_0)\rangle$ are parallel to the two tangent lines respectively, so the normal vector is parallel to

$$\langle 1,0,\tfrac{\partial f}{\partial x}(x_0,y_0)\rangle \times \langle 0,1,\tfrac{\partial f}{\partial y}(x_0,y_0)\rangle = \langle -\tfrac{\partial f}{\partial x}(x_0,y_0), -\tfrac{\partial f}{\partial y}(x_0,y_0), 1\rangle$$

so we can take $\langle a,b,c\rangle = \langle -\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\rangle$

the equation of the tangent plane is

$$-\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)-\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)+1\cdot(z-z_0)=0$$

i.e. $$z-z_0=\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

Example Find the tangent plane to the elliptic paraboloid $z=2x^2+y^2$ at $(1,1,3)$.

let $f(x,y)=2x^2+y^2$, then $\frac{\partial f}{\partial x}=4x$, $\frac{\partial f}{\partial y}=2y$ so $\frac{\partial f}{\partial x}(1,1)=4$ $\frac{\partial f}{\partial y}(1,1)=2$

The tangent plane is $z-3=4(x-1)+2(y-1)$

Recall the linear approximation of a single-variable function at $x=x_0$ is to use the tangent line of the function at $x=x_0$ to approximate the original function near x_0. Now for a two-variable function $f(x,y)$ we use the tangent plane of f at (x_0,y_0) to approximate $f(x,y)$ near (x_0,y_0), and this is the linear approximation for a two-variable function

Example. We can approximate $f(x,y)=2x^2+y^2$ by $z=3+4(x-1)+2(y-1)$ near $(1,1)$, which is the tangent plane we just computed.

i.e. $f(x,y) \approx 3+4(x-1)+2(y-1)$

More generally, $f(x,y) \approx f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$ for (x,y) near (x_0,y_0), under some assumptions for f.

Now we'll figure out the "some assumption" by making the following definition:

If $z=f(x,y)$, then f is differentiable at (x_0,y_0) if

$$f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)=\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$$

where: $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y)\to 0$

When a function is differentiable, we see when Δx and Δy are small, the function and its linear approximation are close to each other since ε_1, ε_2 are also small.

The following theorem can help to tell when a function is differentiable:

Theorem. If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist near (x_0,y_0) and are continuous at (x_0,y_0), then f is differentiable at (x_0,y_0)

Example. Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$, and use its linear approximation to estimate $f(1.1, -0.1)$.

$$\begin{cases} \frac{\partial f}{\partial x} = e^{xy} + xy\,e^{xy} \\ \frac{\partial f}{\partial y} = x^2 e^{xy} \end{cases} \Rightarrow \begin{cases} \frac{\partial f}{\partial x}(1,0) = 1 \\ \frac{\partial f}{\partial y}(1,0) = 1 \end{cases}$$

We see both partial derivatives exist near $(1,0)$ and continuous at $(1,0)$, so f is differentiable at $(1,0)$.

The linear approximation is

$$f(x,y) \approx f(1,0) + \frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)(y-1)$$

$$= 1 + 1\cdot(x-1) + 1\cdot(y-0)$$

$$= x + y$$

so $f(1.1, -0.1) \approx 1.1 - 0.1 = 1$.

If $z = f(x,y)$, we define the differential $dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

We can interpret dx and dy to be the change in x and y respectively, then dz will be the change in z for the tangent plane (i.e the linear approximation), so dz is an approximation of the change in $f(x,y)$.

Example. $f(x,y) = xe^{xy}$, $dz = df = (e^{xy} + xy\,e^{xy})dx + (x^2 e^{xy})dy$.

If x changes from 1 to 1.01, y changes from 0 to -0.02, then

$$dz = \frac{\partial f}{\partial x}(1,0)dx + \frac{\partial f}{\partial y}(1,0)dy = 1\cdot(1.01-1) + 1\cdot(-0.02-0) = -0.01$$

so $f(1.01, -0.02) - f(1,0) \approx dz = -0.01$