MULTI-VARIABLE FUNCTIONS

A function f of n variables is a rule that assigns a unique real number to each ordered n numbers $(x_1, x_2, \dots, x_n)$ in a subset $D \subseteq \mathbb{R}^n$.
We call D the domain of the function.

We'll concentrate on functions of two variables first, and many of the discussions can be naturally extended to n-variable case.

We usually write (x, y) to represent an element in the domain for a 2-variable function, so the function is written as $f(x, y)$.

Example. $z = f(x, y) = \sqrt{x^2 + y^2}$, this is the function that sends (x, y) to the distance between (x, y) and $(0, 0)$.

Example. Ohm's Law

$I = \frac{V}{R}$ The current through a conductor between two points is the quotient of the voltage across the two points and the resistance between the two points.

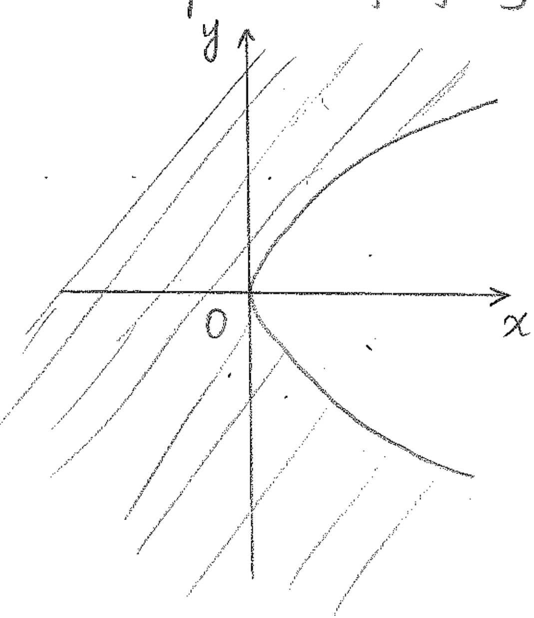
If the domain of a function $f(x,y)$ is not specified, we assume the domain is the largest subset of $\mathbb{R}^2$ that makes the expression of $f(x,y)$ meaningful.

Example. $f(x, y) = x \ln(y^2 - x)$

We see $y^2 - x > 0$, i.e. $y^2 > x$

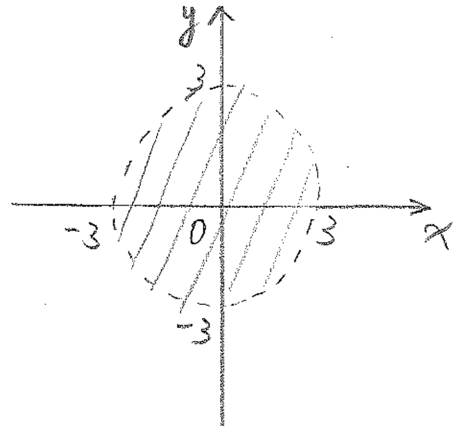
So the domain is

$\{(x, y) \in \mathbb{R}^2 \mid y^2 > x\}$



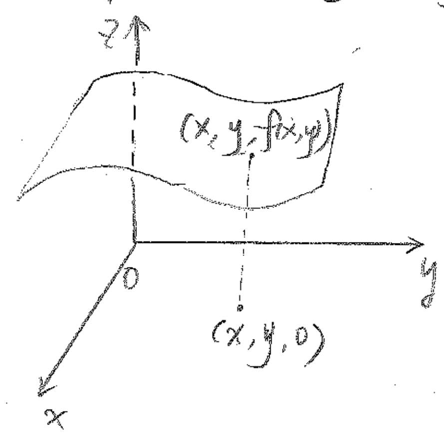
Example. Find the domain of $f(x,y) = \frac{1}{\sqrt{9-x^2-y^2}}$

$9-x^2-y^2>0 \Rightarrow x^2+y^2<9$ so the domain is $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<9\}$

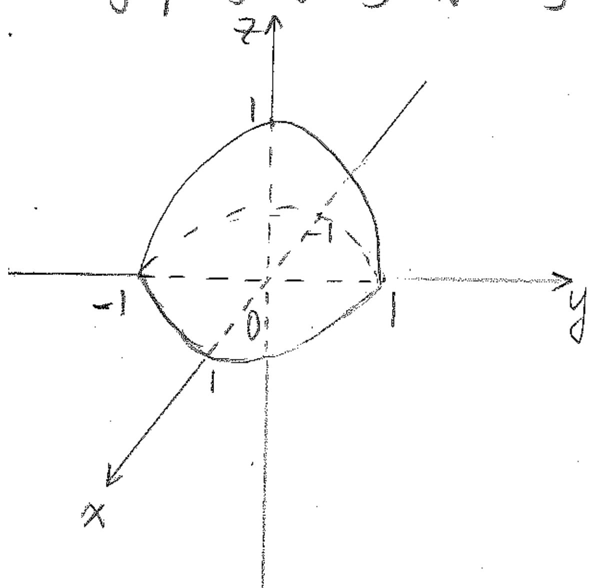


Graph of $z=f(x,y)$:

The graph of the function $f(x,y)$ is the set of all points $(x,y,f(x,y))$ in the 3-dimensional coordinate system.



Example. The graph of $f(x,y)=\sqrt{1-x^2-y^2}$.



Level Curve:

There is one way to illustrate a 2-variable function $f(x,y)$ just on a plane: we make a partition of the plane into disjoint subset, each is a preimage set $f^{-1}(c) = \{(x,y)\in\mathbb{R}^2 \mid f(x,y)=c\}$, which is usually a curve, called the level curve of c, where c is a constant.

Example. $f(x,y) = \sqrt{x^2+y^2}$

The level curve of a constant c ($c>0$) is the set of points $\{(x,y)\in\mathbb{R}^2 \mid \sqrt{x^2+y^2} = c\} = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=c^2\}$. So it's the circle centered at origin with radius c. When $c=0$, the corresponding level curve is just the origin.

Level Surface:

When $f(x,y,z)$ is a 3-variable function, a preimage set $f^{-1}(c) = \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z)=c\}$ is usually a surface instead of a curve. In this case, we call it a level surface

Example. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

The level surface of a constant c ($c>0$) is the set of points $\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{x^2+y^2+z^2} = c\} = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=c^2\}$. So it's the sphere centered at origin with radius c. When $c=0$, the corresponding level surface is just the origin.

LIMITS AND CONTINUITY

We define the limit of $f(x,y)$ as (x,y) approaches (a,b) is L if $\forall\, \varepsilon>0,\ \exists\, \delta>0$ such that $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$ and $(x,y)\in \text{Domain}(f)$

$$\Rightarrow |f(x,y)-f(a,b)|<\varepsilon$$

We will write $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$ in such case.

Note that there are infinitely many ways for (x,y) to approach (a,b). If $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$, then no matter along which way (x,y) approaches (a,b) the corresponding $f(x,y)$ should always approach L.

So we have the following theorem:

Theorem. If $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, C is arbitrary paths approaching (a,b), then $f(x,y)$ will approach $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ along C.

Corollary. If $f(x,y)$ approaches L_1 as (x,y) approaches (a,b) along C_1, and $f(x,y)$ approaches L_2 as (x,y) approaches (a,b) along C_2 (where C_1, C_2 are two paths approaching (a,b)), and $L_1\neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ doesn't exist.

First approach $(0,0)$ along the positive x-axis:

$f(x,0)=\dfrac{x^2-0^2}{x^2+0^2}=1$, so $f(x,y)\to 1$ if $(x,y)\to(0,0)$ along positive x-axis

Next approach $(0,0)$ along the positive y-axis:

$f(0,y)=\dfrac{0^2-y^2}{0^2+y^2}=-1$. So $f(x,y)\to -1$ if $(x,y)\to(0,0)$ along positive y-axis.

So the limit does not exist.

Example. If $f(x,y) = \frac{xy^2}{x^2+y^4}$, show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

First, let $(x,y) \longrightarrow (0,0)$ along positive x-axis, we see

$$f(x,0) = \frac{x \cdot 0^2}{x^2+0^4} = 0. \text{ so } f(x,y) \longrightarrow 0 \text{ along this curve.}$$

Next, let $(x,y) \to (0,0)$ along the curve $x = y^2$ $(y>0)$:

$$f(x,y) = f(y^2,y) = \frac{y^2 \cdot y^2}{(y^2)^2+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}.$$

So $f(x,y) \to \frac{1}{2}$ along this curve.

We see $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example. Show that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exists.

If the limit exists, then it must be 0, since the function approaches 0 along positive x-axis. So we'll try to show $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

For any given $\varepsilon > 0$. we take $\delta = \frac{\varepsilon}{3}$, then

for any (x,y) s.t. $0 < \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{3}$.

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \frac{3x^2|y|}{x^2+y^2} \leq \frac{3(x^2+y^2)|y|}{x^2+y^2} = 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2} < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

So $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Once we defined the limit of $f(x,y)$, we can talk about continuity

A function $f(x,y)$ is called continuous at $(a,b) \in \mathbb{R}^2$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
We say f is continuous on D if f is continuous at every point (a,b) in D

Example. The function $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous:

When $(a,b) \neq (0,0)$, $f(x,y)$ is a rational function near (a,b)

so $f(x,y)$ is continuous at (a,b).

When $(a,b) = (0,0)$ $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. so $f(x,y)$ is continuous at $(0,0)$

Example. The function $f(x,y) = \begin{cases} \frac{x^2+y^2}{x^2-y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous

since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

PARTIAL DERIVATIVES

Recall that for a single-variable function, $f(x)$, it's derivative measures the rate of change of $y=f(x)$ at x.

For a multi-variable function, we can have an analogue of that, but we need to specify the rate of change is with respect to which variable, since we have more than one variables

If $z=f(x,y)$ is a function with 2 variables we define the partial derivatives

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

In other words, $\frac{\partial f}{\partial x}$ is to take derivative of f w.r.t. x, considering y as a constant; $\frac{\partial f}{\partial y}$ is to take derivative of f w.r.t. y, considering x as a constant.

Example $f(x,y)=e^{xy}\ln(x+y)$.

$$\frac{\partial f}{\partial x}=ye^{xy}\ln(x+y)+e^{xy}\cdot\frac{1}{x+y}=e^{x+y}\left(y\ln(x+y)+\frac{1}{x+y}\right)$$

$$\frac{\partial f}{\partial y}=xe^{xy}\ln(x+y)+e^{xy}\cdot\frac{1}{x+y}=e^{xy}\left(x\ln(x+y)+\frac{1}{x+y}\right)$$

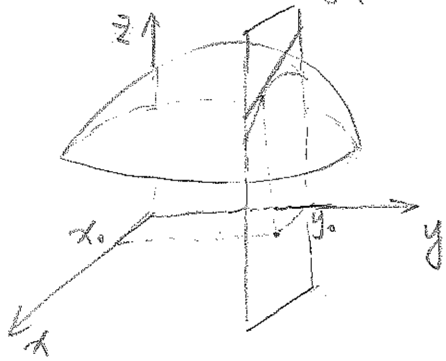
Example. $f(x,y)=\sin\left(\frac{x}{1+y}\right)$.

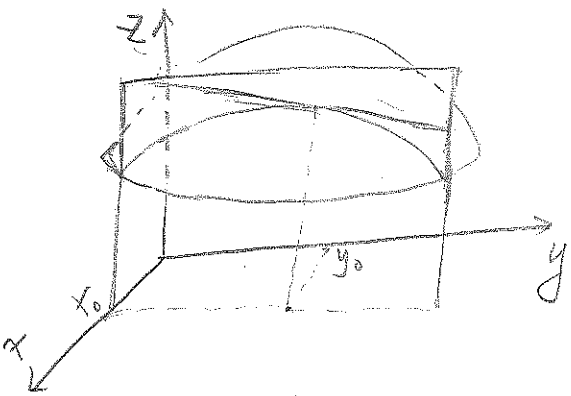
$$\frac{\partial f}{\partial x}=\left(\cos\frac{x}{1+y}\right)\cdot\frac{1}{1+y} \qquad \frac{\partial f}{\partial y}=\left(\cos\frac{x}{1+y}\right)\cdot\left(-\frac{x}{(1+y)^2}\right)=-\left(\cos\frac{x}{1+y}\right)\cdot\frac{x}{(1+y)^2}$$

We can interpret partial derivatives geometrically as the slope of certain curve in the following way:

$z=f(x,y)$ has a graph in the xyz-coordinate system, which consists of points of form $(x,y,f(x,y))$.

At (x_0, y_0), consider the vertical plane $y = y_0$, which intersects the graph of $z = f(x,y)$ at a curve, then the slope of the curve at $x = x_0$ in the plane $y = y_0$ is $\frac{\partial f}{\partial x}(x_0, y_0)$.





Similarly, consider the vertical plane $x = x_0$, which intersects the graph of $z = f(x,y)$ at a curve, then the slope of the curve at $y = y_0$ in the plane $x = x_0$ is $\frac{\partial f}{\partial y}(x_0, y_0)$

Higher Order Derivatives:

If we take the partial derivative of a partial derivative, we will get second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Example. $f(x,y) = x \sin y$

$$\frac{\partial f}{\partial x} = \sin y, \quad \frac{\partial f}{\partial y} = x \cos y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\sin y) = 0.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\sin y) = \cos y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x \cos y) = \cos y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x \cos y) = -x \sin x$$

Observe in the previous example, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. This is not a coincidence, but instead, it's implied by the following theorem:

Clairaut's Theorem: f is defined on a disk that contains the point (a, b), If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For higher order partial derivatives, we can define in a similar way:

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right), \quad \frac{\partial^4 f}{\partial x \partial y^2 \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\right)$$

For functions with more than two variables, we can also define partial derivatives and higher order partial derivatives in a same fashion:

For example, if $f(x, y, z) = x^2 y + y^2 z + z^2 x$ then:

$$\frac{\partial f}{\partial x} = 2xy + z^2 \quad \frac{\partial f}{\partial y} = x^2 + 2yz \quad \frac{\partial f}{\partial z} = y^2 + 2zx$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 2z$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2z, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 2y$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 2z, \quad \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 2y, \quad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 2x$$

The Clairaut's Theorem can also be extended to more variables and higher order partial derivatives:

$f(x_1, x_2, \dots, x_n)$ is defined on a n-dimensional ball centered at $(a_1, a_2, \dots, a_n)$, If all the n-th order partial derivatives are continuous, then we can switch the order of different partial derivatives for an n-th partial derivative