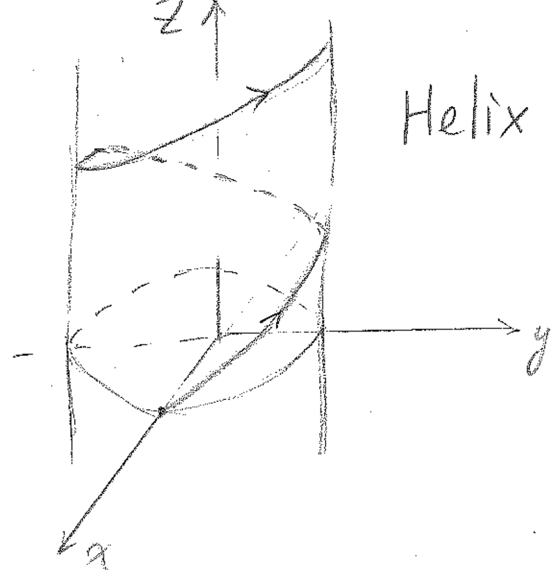
VECTOR FUNCTIONS AND SPACE CURVES

We define a Vactor Function to be a rule that assignes a vector to each real number in the domain. If we use coordinates to represent vectors, we can write the function to be

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle ,$$

where f, g, h are real-value functions.

Example. $r(t) = \langle \cos t, \sin t, t \rangle$



We define the limit of a vector function $\vec{r}(t)$ at $t = t_0$ to be

$$\lim_{t \to t_0} \vec{r}(t) = \langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \rangle$$

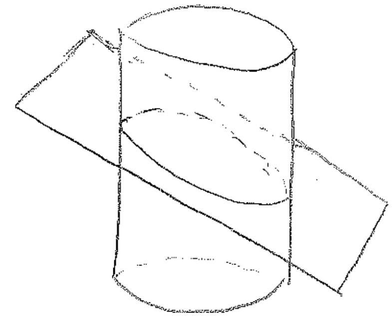
And we say $\vec{r}(t)$ is continuous at t_0 if $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$

In other words, $\vec{r}(t)$ is continuous if and only if all of its component functions are continuous.

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a continuous vector-valued function, We say the corresponding set $\{(x, y, z) \in \mathbb{R}^3 \mid \exists\, t \text{ such that } x = f(t), y = g(t), z = h(t)\}$ is a space curve.

$x = f(t)$, $y = g(t)$, $z = h(t)$. is called the parametric equations for the space curve, and t is called the parameter.

Example. Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$



The projection on the x-y plane of the intersection is the unit circle

so we can let $x=\cos t$, $y=\sin t$. $0 \le t < 2\pi$

then $z=2-y=2-\sin t$, so the intersection curve can be represented by $\vec{r}(t) = \langle \cos t, \sin t, 2-\sin t \rangle$. $t \in [0, 2\pi)$

Since we can define the concept of limit for vector-valued functions, we can also try to define the derivative:

define $\vec{r}\,'(t) = \lim\limits_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$

The vector $\vec{r}\,'(t_0)$ is called the tangent vector of the space curve represented by $\vec{r}(t)$ at the point $\vec{r}(t_0)$, and we define the unit tangent vector to be $\vec{T}(t_0) = \dfrac{\vec{r}\,'(t_0)}{|\vec{r}\,'(t_0)|}$. We define the tangent line of the curve $\vec{r}(t)$ at $\vec{r}(t_0)$ to be the straight line passing through $\vec{r}(t_0)$ and parallel to $\vec{r}\,'(t_0)$.

Example. Find parametric equations for the tangent line to the helix with equation $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$ at $(0, 1, \frac{\pi}{2})$

$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$, and we see $\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle$

so $\vec{r}\,'(\frac{\pi}{2}) = \langle -2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle = \langle -2, 0, 1 \rangle$.

So the parametric equation is $(-2t, 1, \frac{\pi}{2}+t)$. $t \in \mathbb{R}$

There are some rules of differentiation:

<u>Theorem</u> If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, and $f(t)$ is a real-valued differentiable function, $c \in \mathbb{R}$, then:

① $(\vec{u}(t)+\vec{v}(t))' = \vec{u}'(t)+\vec{v}'(t)$

② $(c\vec{u}(t))' = c\vec{u}'(t)$

③ $(f(t)\vec{u}(t))' = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

④ $(\vec{u}(t)\cdot\vec{v}(t))' = \vec{u}'(t)\vec{v}(t) + \vec{u}(t)\vec{v}'(t)$

⑤ $(\vec{u}(t)\times\vec{v}(t))' = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

⑥ $u(f(t)) = f'(t)\vec{u}'(f(t))$

Basically, they can be proved by the fact $\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$

Example If $|\vec{r}(t)| = c$, a constant, prove $\vec{r}'(t) \perp \vec{r}(t)$

$$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$$

Taking derivatives. $\vec{r}'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}'(t) = 0 \Rightarrow 2\vec{r}'(t)\cdot\vec{r}(t) = 0$

$$\Rightarrow \vec{r}'(t)\cdot\vec{r}(t) = 0$$

$$\Rightarrow \vec{r}'(t) \perp \vec{r}(t)$$

We can also define integration of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

Also, we can define the antiderivative (indefinite integral):

$\int \vec{r}(t)\,dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle + \vec{C}$ where $\vec{C}$ is a constant vector.

ARC LENGTH

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a differentiable space curve, we define the length of the curve from $t=a$ to $t=b$ to be the integration:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt.$$

Note that the tangent vector at t is $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$.
So we can rewrite the above formula to be

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

How do we interpret this definition? Recall the definition of Riemann Integral:

$$L = \int_a^b |\vec{r}\,'(t)|\,dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} |\vec{r}\,'(t_i^*)| \Delta t_i \quad , \text{ where } t_i^* \in [t_i, t_{i+1}]$$

When Δt_i is small, we can approximate $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ on $t \in [t_i, t_{i+1}]$ by the corresponding part of straight line

$$l_i = \big(f(t_i) + f'(t_i)(t-t_i),\ g(t_i) + g'(t_i)(t-t_i),\ h(t_i) + h'(t_i)(t-t_i)\big),\ t \in [t_i, t_{i+1}]$$

Then the vector whose two endpoints are the endpoints of this segment is $\langle f'(t_i)(t_{i+1}-t_i),\ g'(t_i)(t_{i+1}-t_i),\ h'(t_i)(t_{i+1}-t_i) \rangle = \langle f'(t_i), g'(t_i), h'(t_i) \rangle \Delta t_i$.
We can take $t_i^* = t_i$, then the vector becomes $\langle f'(t_i^*), g'(t_i^*), h'(t_i^*) \rangle \Delta t_i$ and its length is just $|\vec{r}\,'(t_i^*)| \Delta t_i$. Then the length of the curve is approximated by the sum of all these segments, i.e. $\sum_{i=1}^{n} |\vec{r}\,'(t_i^*)| \Delta t_i$.
As the division into smaller parts is finer and finer, i.e. $\max \Delta t_i \to 0$, This approximation will become better and better. So we define

$$L = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} |\vec{r}\,'(t_i^*)| \Delta t_i = \int_a^b |\vec{r}\,'(t)|\, dt$$

Example. Find the length of the arc of the circular helix with vector equation

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from $(1,0,0)$ to $(1,0,2\pi)$

Observe that $\vec{r}(0) = \langle 1,0,0 \rangle$, $\vec{r}(2\pi) = \langle 1,0,2\pi \rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$, so $|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$

The length of the arc is $\int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$.

We may have noticed that a same segment of curve in space may be parametrized in different ways. If u is a function of t, say $u = \phi(t)$, then $\vec{r}(u)$, $u \in [\phi(a), \phi(b)]$ and $\vec{r}(\phi(t))$, $t \in [a,b]$ give the same curve in space. So we should verify the different parametrizations shall give the same arc length: (We may assume $\phi'(t) > 0$)

$$\int_a^b |\vec{r}'(u(t))| dt = \int_a^b |\phi'(t) \vec{r}'(u)| dt = \int_a^b |\vec{r}'(u)| \phi'(t) dt = \int_a^b |\vec{r}'(u)| d\phi(t)$$
$$= \int_{\phi(a)}^{\phi(b)} |\vec{r}'(u)| du$$

So we see the arc length is independent of parametrizations

Now we introduce a natural parametrization called arc length parametrization

define $s(t) = \int_a^t |\vec{r}'(u)| du$, so $s(t)$ stands for the length of the arc between $\vec{r}(a)$ and $\vec{r}(t)$. The Fundamental Theorem of Calculus implies $\frac{ds}{dt} = |\vec{r}'(t)|$.

We can use s to reparametrize a curve $\vec{r}(t)$:

Example. Reparameterize $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ with respect to arc length measured from $(1,0,0)$ in the direction of increasing t.

Note $\vec{r}'(0) = \langle 1,0,0 \rangle$ so the arc length starts at this point:

$\frac{ds}{dt} = |\vec{r}'(t)| = \sqrt{2} \Rightarrow s = \int_0^t |\vec{r}'(u)| du = \sqrt{2}t \Rightarrow t = \frac{s}{\sqrt{2}}$

So $\vec{r}(s) = \langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$