VECTOR FUNCTIONS AND SPACE CURVES

We define a Vector Function to be a rule that assigns a vector to each real number in the domain. If we use coordinates to represent vectors, we can write the function to be

\[ \vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \]

where \( f, g, h \) are real-value functions.

Example. \( \vec{r}(t) = \langle \cos t, \sin t, t \rangle \)

We define the limit of a vector function \( \vec{r}(t) \) as \( t \to t_0 \) to be

\[ \lim_{t \to t_0} \vec{r}(t) = \langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \rangle \]

And we say \( \vec{r}(t) \) is continuous at \( t_0 \) if

\[ \lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0) \]

In other words, \( \vec{r}(t) \) is continuous if and only if all of its component functions are continuous.

If \( \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \) is a continuous vector-valued function, we say the corresponding set

\[ \{ (x, y, z) \in \mathbb{R}^3 \mid \exists t \text{ such that } x = f(t), \ y = g(t), \ z = h(t) \} \]

is a space curve.

\( x = f(t), \ y = g(t), \ z = h(t) \), is called the parametric equations for the space curve, and \( t \) is called the parameter.
Example. Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.

The projection on the $x$-$y$ plane of the intersection is the unit circle. So we can let \( x = \cos t, y = \sin t, 0 \leq t < 2\pi \) then \( z = 2 - y = 2 - \sin t \), so the intersection curve can be represented by \( \vec{r}(t) = (\cos t, \sin t, 2 - \sin t), t \in [0, 2\pi) \).

Since we can define the concept of limit for vector-valued functions, we can also try to define the derivative:

\[
\vec{r}'(t) = \lim_{{h \to 0}} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = (f'(t), g'(t), h'(t))
\]

The vector \( \vec{r}'(t) \) is called the tangent vector of the space curve represented by \( \vec{r}(t) \) at the point \( \vec{r}(t) \), and we define the unit tangent vector to be \( \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \). We define the tangent line of the curve \( \vec{r}(t) \) at \( \vec{r}(t_0) \) to be the straight line passing through \( \vec{r}(t_0) \) and parallel to \( \vec{r}'(t_0) \).

Example. Find parametric equations for the tangent line to the helix with equation \( \vec{r}(t) = (2\cos t, \sin t, t) \) at \( (0, 1, \frac{\pi}{2}) \).

\[\vec{r}'(t) = (-2\sin t, \cos t, 1)\], and we see \( \vec{r}(\frac{\pi}{2}) = (0, 1, \frac{\pi}{2}) \).

So \( \vec{r}'(\frac{\pi}{2}) = (-2\sin \frac{\pi}{2}, \cos \frac{\pi}{2}, 1) = (-2, 0, 1) \).

So the parametric equation is \((-2t, 1, \frac{\pi}{2}+t)\), \( t \in \mathbb{R} \). 

\( \Box \)
There are some rules of differentiation:

**Theorem.** If \( \vec{u}(t) \) and \( \vec{v}(t) \) are differentiable vector functions, and \( f(t) \) is a real-valued differentiable function, \( c \in \mathbb{R} \), then:

1. \((\vec{u}(t)+\vec{v}(t))' = \vec{u}'(t)+\vec{v}'(t)\)
2. \((c\vec{u}(t))' = c\vec{u}'(t)\)
3. \((f(t)\vec{u}(t))' = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)\)
4. \((\vec{u}(t)\cdot\vec{v}(t))' = \vec{u}'(t)\vec{v}(t) + \vec{u}(t)\vec{v}'(t)\)
5. \((\vec{u}(t)\times\vec{v}(t))' = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)\)
6. \(\vec{u}(-f(t)) = f'(t)\vec{u}'(f(t))\)

Basically, they can be proved by the fact \( \vec{r}(t) = <f(t), g(t), h(t)> \)

**Example.** If \(|\vec{r}(t)| = c \), a constant, prove \( \vec{r}'(t) \perp \vec{r}(t) \)

\[ \vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2 \]

Taking derivatives, \( \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0 \Rightarrow 0 = \vec{r}'(t) \cdot \vec{r}(t) \Rightarrow \vec{r}'(t) \perp \vec{r}(t) \)

We can also define integration of \( \vec{r}(t) = <f(t), g(t), h(t)> \)

\[ \int_a^b \vec{r}(t) \, dt = < \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt > \]

Also, we can define the antiderivative (indefinite integral):

\[ \int \vec{r}(t) \, dt = < \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt > + \vec{c} \text{, where } \vec{c} \text{ is a constant vector.} \]
If \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) is a differentiable space curve, we define the length of the curve from \( t=a \) to \( t=b \) to be the integration:

\[
L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2 + \left(h'(t)\right)^2} \, dt.
\]

Note that the tangent vector at \( t \) is \( \mathbf{T}(t) = \langle f'(t), g'(t), h'(t) \rangle \). So we can rewrite the above formula to be

\[
L = \int_a^b |\mathbf{T}(t)| \, dt.
\]

How do we interpret this definition? Recall the definition of Riemann Integral:

\[
L = \int_a^b |\mathbf{T}(t)| \, dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^b |\mathbf{T}(t_i^*)| \Delta t_i,
\]

where \( t_i^* \in [t_i, t_{i+1}] \).

When \( \Delta t_i \) is small, we can approximate \( \mathbf{T}(t) = \langle f(t), g(t), h(t) \rangle \) on \( t \in [t_i, t_{i+1}] \) by the corresponding part of straight line

\[
\mathbf{l}_{\mathbf{i}} = \langle f(t_i) + f'(t_i)(t - t_i), g(t_i) + g'(t_i)(t - t_i), h(t_i) + h'(t_i)(t - t_i) \rangle, \quad t \in [t_i, t_{i+1}].
\]

Then the vector whose two endpoints are the endpoints of this segment is \( \langle f'(t_i)(t_{i+1} - t_i), g'(t_i)(t_{i+1} - t_i), h'(t_i)(t_{i+1} - t_i) \rangle = <f'(t_i), g'(t_i), h'(t_i)> \Delta t_i \).

We can take \( t_i^* = t_i \), then the vector becomes \( <f'(t_i^*), g'(t_i^*), h'(t_i^*)> \Delta t_i \) and its length is just \( |\mathbf{T}(t_i^*)| \Delta t_i \). Then the length of the curve is approximated by the sum of all these segments, i.e. \( \sum_{i=1}^{b} |\mathbf{T}(t_i^*)| \Delta t_i \).

As the division into smaller parts is finer and finer, i.e. \( \max \Delta t_i \to 0 \), this approximation will become better and better. So we define

\[
L = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^b |\mathbf{T}(t_i^*)| \Delta t_i = \int_a^b |\mathbf{T}(t)| \, dt.
\]
Example. Find the length of the arc of the circular helix with vector equation 
\[ \mathbf{r}(t) = <\cos t, \sin t, t> \text{ from } (1,0,0) \text{ to } (1,0,2\pi) \]

Observe that \[ \mathbf{r}'(0) = <1,0,0> , \mathbf{r}'(2\pi) = <1,0,2\pi> \]

\[ \mathbf{r}'(t) = <-\sin t, \cos t, 1> , \text{ so } |\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2} \]

The length of the arc is \[ \int_0^{2\pi} |\mathbf{r}'(t)| \, dt = 2\sqrt{2} \pi \]

We may have noticed that a same segment of curve in space may be parametrized in different ways. If \( u \) is a function of \( t \), say \( u = \varphi(t) \), then \( \mathbf{r}(u), u \in [\varphi(a), \varphi(b)] \) and \( \mathbf{r}(\varphi(t)), t \in [a,b] \) give the same curve in space. So we should verify the different parametrizations shall give the same arc length. (We may assume \( \varphi'(t) > 0 \))

\[ \int_a^b |\mathbf{r}'(u(t))| \, dt = \int_a^b |\varphi'(t)| \mathbf{r}'(u) \, |dt| = \int_a^b |\mathbf{r}'(u)| |\varphi'(t)| \, dt = \int_a^b |\mathbf{r}'(u)| |\varphi'(u)| \, du = \int_{\varphi(a)}^{\varphi(b)} |\mathbf{r}'(u)| \, du \]

So we see the arc length is independent of parametrizations.

Now we introduce a natural parametrization called arc length parametrization. Define \( s(t) = \int_a^t |\mathbf{r}'(u)| \, du \), so \( s(t) \) stands for the length of the arc between \( \mathbf{r}(a) \) and \( \mathbf{r}(t) \). The Fundamental Theorem of Calculus implies \( \frac{ds}{dt} = |\mathbf{r}'(t)| \).

We can use \( s \) to reparametrize a curve \( \mathbf{r}(t) \):

Example. Reparameterize \( \mathbf{r}(t) = <\cos t, \sin t, t> \) with respect to arc length measured from \((1,0,0)\) in the direction of increasing \( t \).

Note \( \mathbf{r}'(0) = <1,0,0> \), so the arc length starts at this point:

\[ \frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2} , \Rightarrow s = \int_0^t |\mathbf{r}'(u)| \, du = \sqrt{2} t \Rightarrow t = \frac{s}{\sqrt{2}} \]

So \( \mathbf{r}(s) = <\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}> \).