LINES AND PLANES

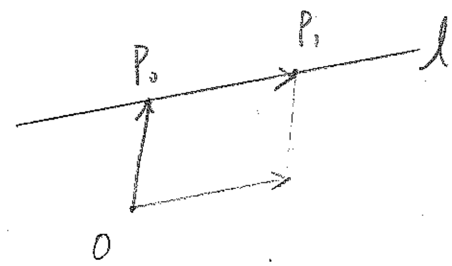
Equation of a Line:

We can make use of vectors to describe a straight line:

We first pick a point $P_0=(x_0,y_0,z_0)$ on the straight line l, then we get the corresponding position vector $\overrightarrow{OP_0}=\langle x_0,y_0,z_0\rangle$.

Take another point $P_1=(x_1,y_1,z_1)$ on the straight line l. the vector $\overrightarrow{P_0P_1}=\langle x_1-x_0,\,y_1-y_0,\,z_1-z_0\rangle$ is parallel to l.

Now for any point $P=(x,y,z)$ on l



$\overrightarrow{P_0P} \parallel \overrightarrow{P_0P_1}$, so $\exists\ \lambda\in\mathbb{R}$ such that

$$\overrightarrow{P_0P}=\lambda\overrightarrow{P_0P_1}$$

$$\overrightarrow{OP}=\overrightarrow{OP_0}+\overrightarrow{P_0P}=\overrightarrow{OP_0}+\lambda\overrightarrow{P_0P_1}=\langle x_0,y_0,z_0\rangle+\lambda\langle x_1-x_0,\,y_1-y_0,\,z_1-z_0\rangle$$

$$=\lambda\langle x_1,y_1,z_1\rangle+(1-\lambda)\langle x_0,y_0,z_0\rangle$$

We obtain the first equation for a straight line:

If l is a straight line passing through $P_0=(x_0,y_0,z_0)$ and $P_1=(x_1,y_1,z_1)$ then it is the set of points of the form

$$(\lambda x_1+(1-\lambda)x_0,\ \lambda y_1+(1-\lambda)y_0,\ \lambda z_1+(1-\lambda)z_0),\quad \lambda\in\mathbb{R}.$$

Sometimes, we would like to write $\overrightarrow{P_0P_1}=\langle a,b,c\rangle$, in such case,

$$\overrightarrow{OP}=\overrightarrow{OP_0}+\lambda\overrightarrow{P_0P_1}=\langle x_0,y_0,z_0\rangle+\lambda\langle a,b,c\rangle=\langle x_0+\lambda a,\ y_0+\lambda b,\ z_0+\lambda c\rangle$$

So we obtain the second equation for a straight line:

If l is a straight line passing through $P_0=(x_0,y_0,z_0)$ and $l\parallel\langle a,b,c\rangle$, then it is the set of points of the form

$$(x_0+\lambda a,\ y_0+\lambda b,\ z_0+\lambda c),\quad \lambda\in\mathbb{R}$$

Example. Find an equation of the line that passing through the points $(1,3,5)$ and $(2,4,6)$

The equation is : $(2\lambda+(1-\lambda)\cdot 1, 4\lambda+(1-\lambda)\cdot 3, 6\lambda+(1-\lambda)\cdot 5)$

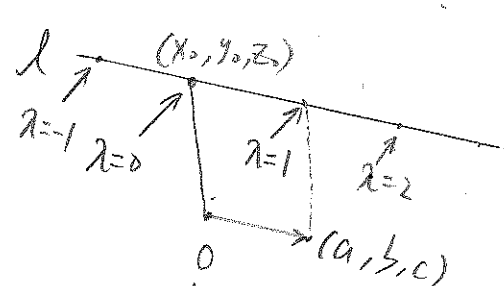
$$= (\lambda+1, \lambda+3, \lambda+5), \quad \lambda \in \mathbb{R}$$

Example. Find an equation of the line that passing through the point $(1,3,5)$ and parallel to $\langle 1,1,1\rangle$.

The equation is $(1+\lambda\cdot 1, 3+\lambda\cdot 1, 5+\lambda\cdot 1) = (1+\lambda, 3+\lambda, 5+\lambda), \lambda\in\mathbb{R}$

An important observation of the equation $(x_0+\lambda a, y_0+\lambda b, z_0+\lambda c)$ is that as λ changes, the corresponding point on l changes in a continuous fashion. when $\lambda=0$, it stands for the point (x_0, y_0, z_0) and when $\lambda>0$, the corresponding point is in "positive" direction with respect to the vector $\langle a,b,c\rangle$; when $\lambda<0$, the corresponding point is in "negative" direction with respect to the vector $\langle a,b,c\rangle$.

So a line segment on l can be obtained by restricting λ on certain interval



In particular, for the line segment between (x_0,y_0,z_0) and (x_1,y_1,z_1) it can be represented by $((1-\lambda)x_0+\lambda x_1, (1-\lambda)y_0+\lambda y_1, (1-\lambda)z_0+\lambda z_1)$ where $0\le\lambda\le 1$

Another form of equation for a straight line is the symmetric equation which can be obtained from the equation $(x_0+\lambda a, y_0+\lambda b, z_0+\lambda c), \lambda \in \mathbb{R}$ by eliminating λ: (when $abc \neq 0$)

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Also, this equation can be understood in another way: it comes from the fact that if $\langle a,b,c\rangle$ is parallel to a line, the for any two points $P_0=(x_0,y_0,z_0)$ and $P=(x,y,z)$ on the line.

$\overrightarrow{P_0P} \parallel \langle a,b,c\rangle$ i.e $\langle x-x_0, y-y_0, z-z_0\rangle \parallel \langle a,b,c\rangle$, so the ratio of each corresponding coordinate are the same.

Example. Show that the lines L_1 and L_2 with parametric equations

L_1: $x=1+t$, $y=-2+3t$, $z=4-t$ $\quad t\in\mathbb{R}$.

L_2: $x=2s$, $y=3+s$, $z=-3+4s$, $\quad s\in\mathbb{R}$

are skew lines, i.e. they do not intersect and are not parallel

They're not parallel since $\langle 1,3,-1\rangle$ and $\langle 2,1,4\rangle$ are not parallel

They do not intersect since the equations

$$\begin{cases} 1+t=2s \\ -2+3t=3+s \\ 4-t=-3+4s \end{cases}$$ has no solution.

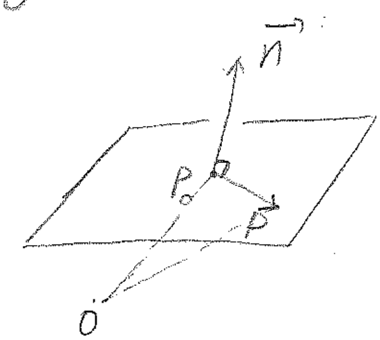
Equation of a Plane:

If S is a plane and $P_0 = (x_0, y_0, z_0) \in S$, then there exists a unique straight line passing through P_0 and perpendicular to S. (Perpendicular to S means it is perpendicular to all the straight lines in S)

We take a vector $\vec{n}$, which is parallel to this straight line, we say $\vec{n}$ is a normal vector of the plane S.

If $P = (x, y, z) \in S$, then $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ is parallel to a straight line contained in S, hence $\overrightarrow{P_0P} \perp \vec{n}$. If we know $\vec{n} = \langle a, b, c \rangle$, then it follows $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

i.e. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$



And if we rearrange the above equation, we find that it can be written in the form of

$ax + by + cz + d = 0$

(where $d = -ax_0 - by_0 - cz_0$)

Example. Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\vec{n} = \langle 2, 3, 4 \rangle$.

The equation is

$$2(x-2) + 3(y-4) + 4(z+1) = 0$$

or you can reduce it to $2x + 3y + 4z - 12 = 0$

Sometimes, we need to find $\vec{n}$ by ourselves based on the known information: A typical case is to find the equation of a plane passing through three given points: P, Q, R. The strategy is to first take $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$, then use this $\vec{n}$ as the normal vector

Example. Find an equation of the plane that passes through $P = \langle 1, 3, 2 \rangle$

$Q = (3, -1, 6)$, $R = (5, 2, 0)$

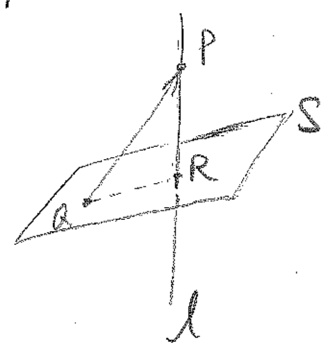
$\overrightarrow{PQ} = \langle 2, -4, 4 \rangle$, $\overrightarrow{PR} = \langle 4, -1, -2 \rangle$.

Then $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 12, 20, 14 \rangle$. so the equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

There are some different types of geometric questions once we know the equations of planes:

① The distance from a point P in space to a plane S:



Constuct the straight line l passing through P and perpendicular to the plane S, and l intersects S at point R. By definition, the length of the segment PR is the distance from P to S. So how to compute this distance?

We pick a point Q on S, and consider the vector $\overrightarrow{QP}$.

If $\vec{n}$ is a normal vector of S, then $\overrightarrow{QP} \cdot \vec{n} = |\overrightarrow{QP}| \cdot |\vec{n}| \cdot \cos\theta$.

where θ is the angle between the two vectors. Observe that

$$|\overrightarrow{PR}| = |\overrightarrow{QP}| \cdot |\cos\theta| = \frac{|\overrightarrow{QP} \cdot \vec{n}|}{|\vec{n}|}$$

Example. Find the distance between the point $P=(1,3,5)$ and the plane

$$2x-3y+7z-9=0$$

From the equation of the plane, we see $\vec{n}=\langle 2,-3,7\rangle$ is a normal vector of the plane.

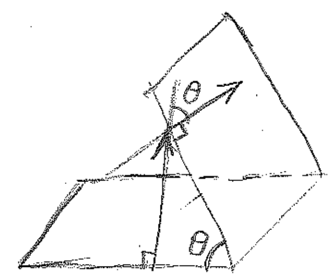
let $x=0$, $z=0$, in the equation of the plane, we get $y=-3$.

so $Q=(0,-3,0)$ is a point on the plane, $\overrightarrow{QP}=\langle 1,6,5\rangle$

so the distance is $\dfrac{|\overrightarrow{QP}\cdot\vec{n}|}{|\vec{n}|}=\dfrac{2\times 1-3\times 6+7\times 5}{\sqrt{2^2+(-3)^2+7^2}}=\dfrac{19}{\sqrt{62}}=\dfrac{19}{62}\sqrt{62}$

② Find the angle between two planes:

We only need to compute the angle between the normal vectors of the two planes.
If the result is an obtuse angle, we take its supplementary angle



Example. Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$

$\vec{n}_1=\langle 1,1,1\rangle$. $\vec{n}_2=\langle 1,-2,3\rangle$

so $\cos\theta=\dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}=\dfrac{1\times 1+1\times(-2)+1\times 3}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{1^2+(-2)^2+3^2}}=\dfrac{2}{\sqrt{42}}=\dfrac{1}{21}\sqrt{42}$

$\theta=\arccos\dfrac{\sqrt{42}}{21}$

③. Find the equation of the intersection of two planes.

We need to first find a point on the intersection line, by using the two equations of the planes. Then we take the cross product of the two normal vectors to get a vector parallel to the intersection line:

Example. Find the equation of the line of intersection l of the planes $x+y+z=1$ and $x-2y+3z=1$

$\vec{n}_1 = \langle 1, 1, 1 \rangle$. $\vec{n}_2 = \langle 1, -2, 3 \rangle$ then

$\vec{n}_1 \times \vec{n}_2 = \langle 5, -2, -3 \rangle$

We also need to find a point on l, which is the intersection of the planes, so the point should satisfy both equations.

By letting $z=0$, we get $\begin{cases} x+y=1 \\ x-2y=1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=0 \end{cases}$.

so $(1, 0, 0) \in l$.

so an equation of l is

$$(1+5\lambda, -2\lambda, -3\lambda), \lambda \in R$$