Cross Product.

If $\vec{u}$ and $\vec{v}$ are 3-dimensional vectors, $\vec{u} = \langle x_1, y_1, z_1 \rangle$, $\vec{v} = \langle x_2, y_2, z_2 \rangle$.

Then define $\vec{u} \times \vec{v} = \langle y_1 z_2 - y_2 z_1,\ x_2 z_1 - x_1 z_2,\ x_1 y_2 - x_2 y_1 \rangle$

There is an easier way to remember the formula for $\vec{u} \times \vec{v}$:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1 z_2 - y_2 z_1)\vec{i} - (x_1 z_2 - x_2 z_1)\vec{j} + (x_1 y_2 - x_2 y_1)\vec{z}$$

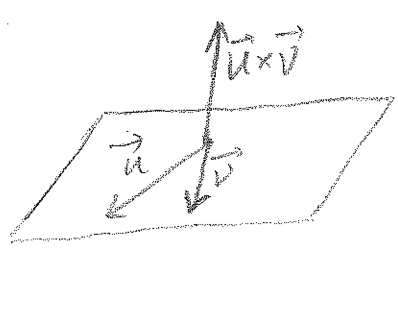
$$= \langle y_1 z_2 - y_2 z_1,\ x_2 z_1 - x_1 z_2,\ x_1 y_2 - x_2 y_1 \rangle$$

Recall that the above expression is the determinant of the 3x3 matrix.

Theorem. $\vec{u} \times \vec{v} \perp \vec{u}$ and $\vec{u} \times \vec{v} \perp \vec{v}$.

Pf. Let $\vec{u} = \langle x_1, y_1, z_1 \rangle$ and $\vec{v} = \langle x_2, y_2, z_2 \rangle$, a direct computation will show the results.

Now we know that $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, so it's perpendicular to the plane spanned by $\vec{u}$ and $\vec{v}$. Now there are still two opposite directions to consider, and we have to figure out which will be the direction for $\vec{u} \times \vec{v}$. By some linear algebra, we can confirm that the direction of $\vec{u} \times \vec{v}$ follows from the so-called "right hand rule." in formal language, it means $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ forms a basis of the space which has the same orientation as the standard basis $\vec{i}, \vec{j}, \vec{k}$



Now we know geometrically how to figure out the direction of $\vec{u} \times \vec{v}$.
The next question is what is the length of $\vec{u} \times \vec{v}$?

Theorem. If θ is the angle between $\vec{u}$ and $\vec{v}$, then $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin\theta$

i.e. The area of the parallelogram spanned by the vectors $\vec{u}$ & $\vec{v}$.

Pf. Let $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$, then

$$|\vec{u} \times \vec{v}|^2 = (y_1 z_2 - y_2 z_1)^2 + (x_2 z_1 - x_1 z_2)^2 + (x_1 y_2 - y_2 x_1)^2$$

$$= y_1^2 z_2^2 + y_2^2 z_1^2 - 2 y_1 y_2 z_1 z_2 + x_2^2 z_1^2 + x_1^2 z_2^2 - 2 x_1 x_2 z_1 z_2$$
$$+ x_1^2 y_2^2 + x_2^2 y_1^2 - 2 x_1 x_2 y_1 y_2$$

$$|\vec{u}|^2 \cdot |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 = (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1 x_2 + y_1 y_2 + z_1 z_2)^2$$

$$= x_1^2 x_2^2 + y_1^2 y_2^2 + z_1^2 z_2^2 + x_1^2 y_2^2 + y_1^2 x_2^2 + x_1^2 z_2^2 + z_1^2 x_2^2 + y_1^2 z_2^2 + z_1^2 y_2^2$$
$$- x_1^2 x_2^2 - y_1^2 y_2^2 - z_1^2 z_2^2 - 2 x_1 x_2 y_1 y_2 - 2 x_1 x_2 z_1 z_2 - 2 y_1 y_2 z_1 z_2$$

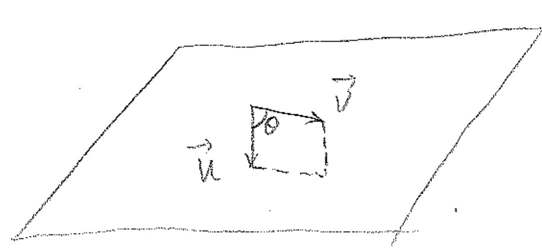
$$= x_1^2 y_2^2 + x_2^2 y_1^2 + x_1^2 z_2^2 + x_2^2 z_1^2 + y_1^2 z_2^2 + y_2^2 z_1^2$$
$$- 2 y_1 y_2 z_1 z_2 - 2 x_1 x_2 z_1 z_2 - 2 x_1 x_2 y_1 y_2$$

So
$$|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$
$$= |\vec{u}|^2 \cdot |\vec{v}|^2 - |\vec{u}|^2 \cdot |\vec{v}|^2 \cos^2\theta$$
$$= |\vec{u}|^2 \cdot |\vec{v}|^2 (1 - \cos^2\theta)$$
$$= |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta.$$

Since $0 \le \theta \le \pi$, $\sin\theta \ge 0$, when we take squareroot on both sides,

$$|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin\theta$$

By the Sine Theorem, $|\vec{u}| \cdot |\vec{v}| \sin\theta$ is exactly the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.



Corollary. Two non-zero vectors are parallel iff their cross product is 0.

Example. $\vec{u} = \langle 1, 3, 4 \rangle$, $\vec{v} = \langle 2, 7, -5 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \vec{k}$$

$$= (3\times(-5) - 7\times 4)\vec{i} - (1\times(-5) - 2\times 4)\vec{j} + (1\times 7 - 2\times 3)\vec{k}$$

$$= -43\vec{i} + 13\vec{j} + 1\vec{k}$$

$$= \langle -43, 13, 1 \rangle$$

Example. Find a vector perpendicular to the plane that passes through the points $P = (1, 4, 6)$, $Q = (-2, 5, -1)$, $R = (1, -1, 1)$

$\overrightarrow{PQ} = \langle -3, 1, -7 \rangle$ and $\overrightarrow{PR} = \langle 0, -5, -5 \rangle$ are on this plane.

So we can choose $\overrightarrow{PQ} \times \overrightarrow{PR}$, which is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, hence perpendicular to the plane.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix} \vec{k}$$

$$= -40\vec{i} - 15\vec{j} + 15\vec{k}$$

$$= \langle -40, -15, 15 \rangle$$

Example. Find the area of the triangle with vertices $(1, 4, 6)$, $(-2, 5, -1)$ and $(1, -1, 1)$.

The area of the triangle is half of the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. So the area of the triangle is

$$\frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \cdot \sqrt{(-40)^2 + (-15)^2 + 15^2} = \frac{5}{2}\sqrt{82}$$

Triple Product.

$\vec{u}, \vec{v}, \vec{w}$ are 3-dimensional vectors. We define the triple product of them to be $\vec{u}\cdot(\vec{v}\times\vec{w})$

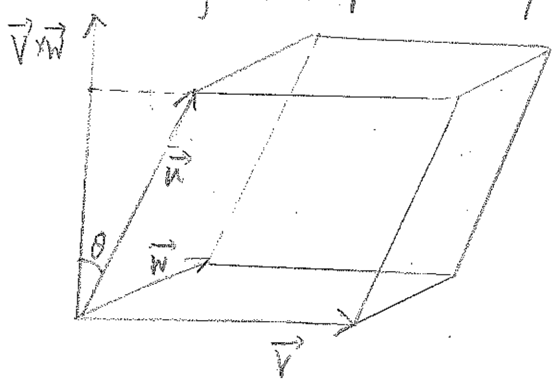
If $\vec{u} = \langle x_1, y_1, z_1\rangle$, $\vec{v} = \langle x_2, y_2, z_2\rangle$, $\vec{w} = \langle x_3, y_3, z_3\rangle$, then

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \langle x_1, y_1, z_1\rangle \cdot \left\langle \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix}, -\begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix}, \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \right\rangle$$

$$= x_1\begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1\begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The absolute value of determinant is just the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.



$$|\vec{u}\cdot(\vec{v}\times\vec{w})| = |\vec{u}|\cdot|\vec{v}\times\vec{w}|\cdot|\cos\theta|$$

$$= \underbrace{\big||\vec{u}|\cos\theta\big|}_{\text{height}} \cdot \underbrace{|\vec{v}\times\vec{w}|}_{\text{area of the base parallelogram}}$$

$$= \text{Volume of the parallelepiped}$$

Note that three vectors are coplanar if and only if the parallelepiped has volume 0, if and only if their triple product is 0