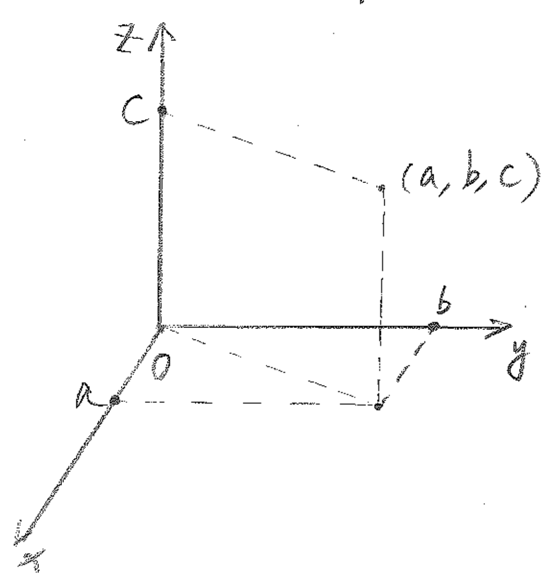
THREE DIMENSIONAL EUCLIDEAN SPACE

We are looking for a way to give description to any point in the 3-dimensional space. Recall that when we try do describe the positon of a point, we use the Cartisian Coordinate System. We shall have an analogue of that, by thinking about the 3-dimensional space as a union of parallel planes, and each real number corresponds to a copy of the plane. Intuitively, we can regard each horizontal plane as a floor, and each floor is labeled with a unique real number, so the 3-dimensional space is a building with uncountably many (the cardinality of $\mathbb{R}$) floors!

In order to specify a location in the 3-dimensional space, we need to specify its floor and it's location projected to the ground. This indicates the following construction:

Take a horizontal plane, and build an x-y Cartisian Coordinate System on it. Now take a real number line, passing through the chosen x-y plane, with the origion of the real line coincides with that of the x-y plane, and the real line is perpendicular to the x-y plane, pointing upward:

We call this real line the z-axis

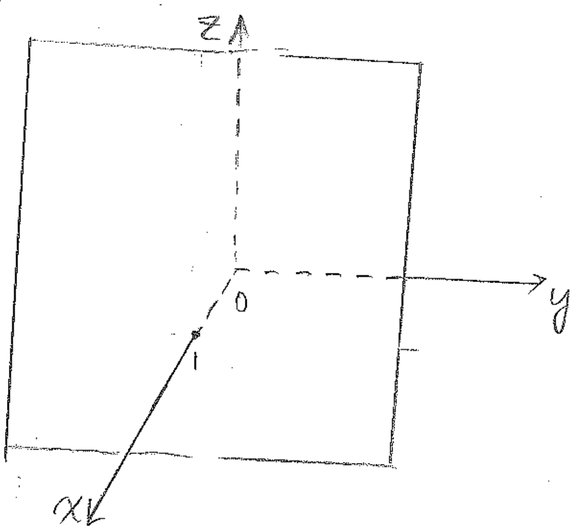


Now points in the 3-dimensional space and ordered numbers (a, b, c) are in one-to-one correspondence.

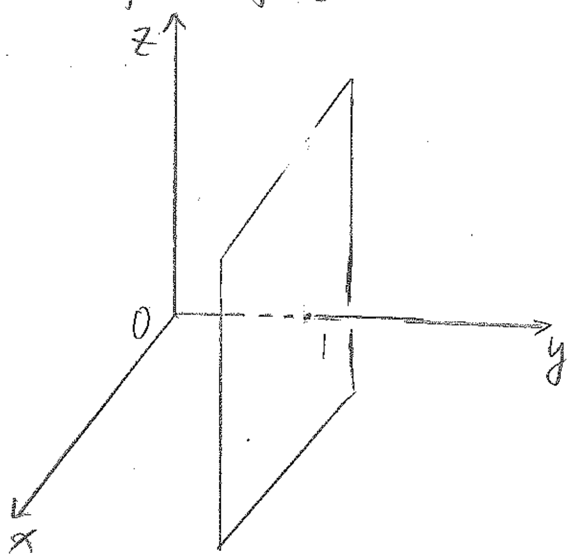
c represents the "altitude" of the point, which is the projection of the point to the z-axis. (a, b) is the coordinate of the projection of the point on the x-y plane.

Similar to the 2-dimension case, in 3-dimensions, an equation involving x, y, z, say $E(x,y,z)=0$, corresponds to a surface in the Cartisian Coordinate Space, which consists of points with coordinate (x,y,z) satisfying $E(x,y,z)=0$

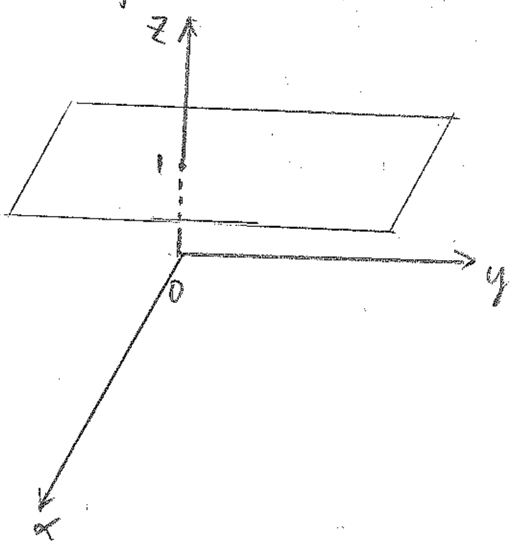
Example. Surface of $x=1$:



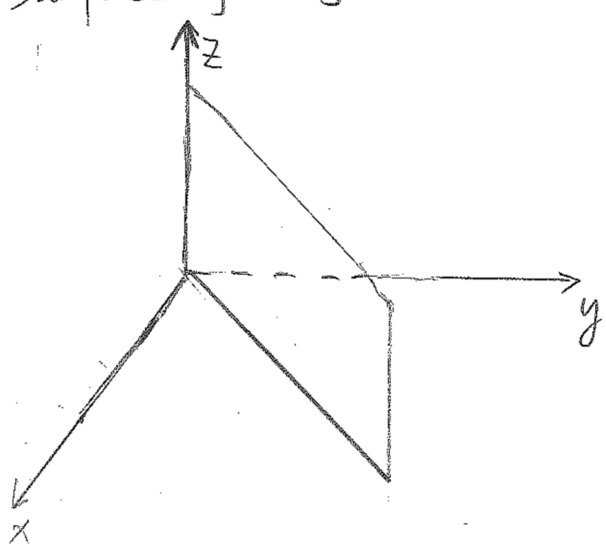
Surface of $y=1$:



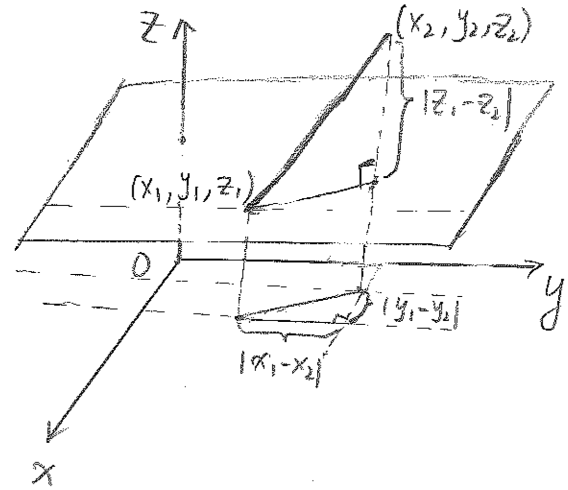
Surface of $z=1$:



Surface of $y=x$



How do we compute the distance of two points in the 3-dimensional coordinate system?



By Pythagorean Theorem, we see the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

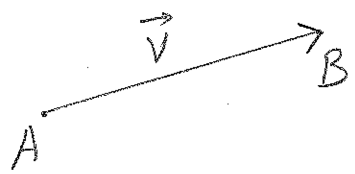
$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2}.$$

With the help of the above formula, we can describe a sphere by an algebraic equation:

For a sphere with center (x_0, y_0, z_0) and radius r, its equation is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

VECTORS

A vector (in the Euclidean Space $\mathbb{R}^3$) is a line segment with a specified direction. It is determined by the length of the line segment (the quantity) and also its direction. The direction of the vector induces an orientation on the line segment, and with respect to this orientation, we can define the initial point and the terminal point to be the two ends of the segment, if imagining a particle travelling on the segment along its given direction.

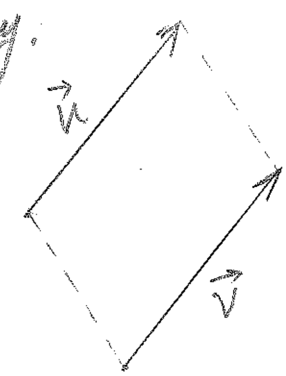


We usually use letters with an arrow above to denote vectors, e.g. $\vec{v}$. In the above figure, A is the initial point of $\vec{v}$, and B is the terminal point of $\vec{v}$. Another way to write this vector is $\overrightarrow{AB}$.

Equivalence of Vectors:

If two vectors $\vec{u}$ and $\vec{v}$ have same length and same direction (possibly with different initial points), the we say the two vectors are equivalent.

We can obtain a vector which will be equivalent to a given vector $\vec{v}$ by translating all the points on $\vec{v}$ in the same way.



Scalar Multiplication.

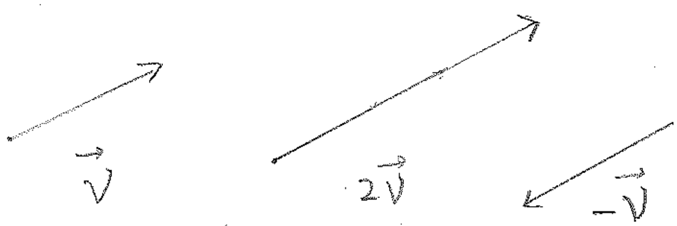
If $\vec{v}$ is a vector and c is a real number (call it a scalar), we can define the scalar multiplication $c\vec{v}$, which is a vector, as follows:

The length of $c\vec{v}$ is $|c|$ times the length of $\vec{v}$, and for the direction:

① If $c>0$, $c\vec{v}$ has the same direction as that of $\vec{v}$.

② If $c=0$, $c\vec{v}=\vec{0}$, no direction

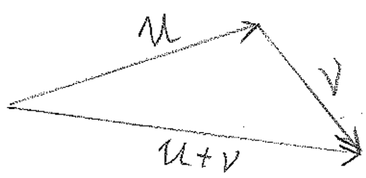
③ If $c<0$, $c\vec{v}$ has the opposite direction as that of $\vec{v}$.

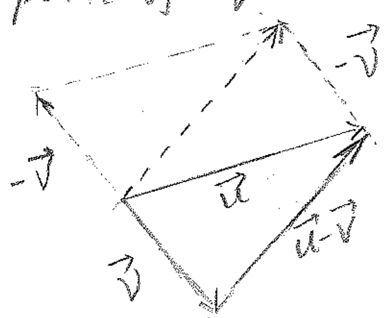


In particular, define the negative of $\vec{v}$ to be $-\vec{v}=(-1)\vec{v}$

Vector Addition.

If $\vec{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.





Vector Subtraction

If $\vec{u}$ and $\vec{v}$ are vectors, then define the subtraction $\vec{u}-\vec{v}$ to be $\vec{u}+(-\vec{v})$. Geometrically, if $\vec{u}$ and $\vec{v}$ are two vectors with a common initial point, then $\vec{u}-\vec{v}$ is the vector from the terminal point of $\vec{v}$ to the terminal point of $\vec{u}$

Vector representations in the Coordinate System.

In a coordinate System, for any point $P=(x,y,z)$, we can construct the vector with initial point the origin O, and terminal point P. $\overrightarrow{OP}$.

We call $\overrightarrow{OP}$ the position vector of the point P.

Observe that the assignment of position vector to point P gives an identification of (3-dimensional) vectors and points in 3-dimensional space.

Another observation is that for each vector $\vec{v}$, and any point $A=(a_1,a_2,a_3)$ there is a point $B=(b_1,b_2,b_3)$ such that $\vec{v}$ is equivalent to $\overrightarrow{AB}$.

We say $\overrightarrow{AB}$ is a representation of the vector $\vec{v}$

In a coordinate system, if $\vec{v}$ is equivalent to the position vector of the point $P=(a,b,c)$, then we denote $\vec{v}$ by $\vec{v}=\langle a,b,c\rangle$.

With the help of the above notation, we can compute vector addition, subtraction and scalar multiplication algebraically.

Theorem. If $\vec{u}=\langle a_1,b_1,c_1\rangle$, $\vec{v}=\langle a_2,b_2,c_2\rangle$, and $\lambda\in\mathbb{R}$, then:

$$\vec{u}+\vec{v}=\langle a_1+a_2,\ b_1+b_2,\ c_1+c_2\rangle,\quad \vec{u}-\vec{v}=\langle a_1-a_2,\ b_1-b_2,\ c_1-c_2\rangle$$

$$\lambda\vec{u}=\langle\lambda a_1,\lambda b_1,\lambda c_1\rangle$$

Also, by using coordinates, it is easy to compute the length of a vector. We usually denote the length of a vector $\vec{v}$ by $|\vec{v}|$.

Theorem. If $\vec{v}=\langle a,b,c\rangle$, then $|\vec{v}|=\sqrt{a^2+b^2+c^2}$.

Proof. If $\vec{v}=\langle a,b,c\rangle$, the $\vec{v}$ is the position vector for the point $P=(a,b,c)$, i.e. $\vec{v}$ is equivalent to the vector with initial point $(0,0,0)$ and terminal point (a,b,c), and the length of this vector is same as the distance between $(0,0,0)$ and (a,b,c), which is $\sqrt{a^2+b^2+c^2}$

Example. If $\vec{u} = \langle 3, 2, 5 \rangle$, $\vec{V} = \langle 4, 1, 3 \rangle$, then compute $\vec{u} - 2\vec{V}$

$$\vec{u} - 2\vec{V} = \langle 3, 2, 5 \rangle - 2 \cdot \langle 4, 1, 3 \rangle =$$

$$= \langle 3, 2, 5 \rangle - \langle 8, 2, 6 \rangle$$

$$= \langle -5, 0, -1 \rangle$$

If a vector has length 1, we call it a unit vector. Some examples are $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$.

If $\vec{V}$ is a nonzero vector, then there is a unit vector which has the same direction as $\vec{V}$, and the unit vector is $\frac{1}{|\vec{V}|}\vec{V}$

Example. Find the unit vector that has the same direction as the vector $\vec{V} = \langle 2, -2, -1 \rangle$.

$|\vec{V}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$ so the unit vector is

$$\frac{1}{|\vec{V}|}\vec{V} = \frac{1}{3}\langle 2, -2, -1 \rangle = \langle \tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{1}{3} \rangle$$

Dot Product:

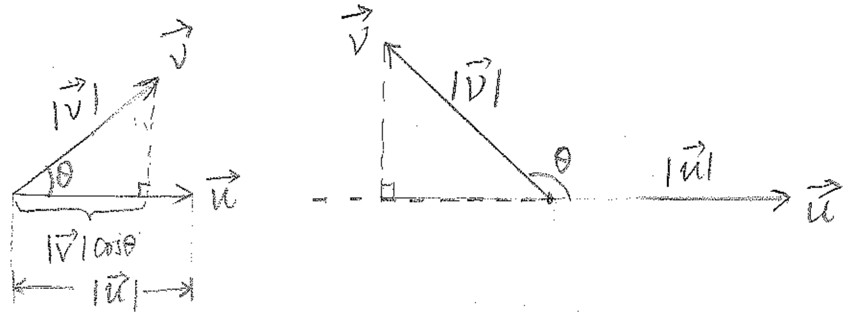
If $\vec{u} = \langle a_1, a_2, a_3 \rangle$ and $\vec{V} = \langle b_1, b_2, b_3 \rangle$ are vectors, define $\vec{u} \cdot \vec{V} = a_1 b_1 + a_2 b_2 + a_3 b_3$, the dot product of $\vec{u}$ and $\vec{V}$.

Theorem. If $\vec{u}, \vec{V}, \vec{w}$ are vectors, then:

(1). $\vec{V} \cdot \vec{V} = |\vec{V}|^2$. (2). $\vec{u} \cdot \vec{V} = \vec{V} \cdot \vec{u}$. (3) $\vec{u} \cdot (\vec{V} + \vec{w}) = \vec{u} \cdot \vec{V} + \vec{u} \cdot \vec{w}$

(4) $(\lambda \vec{u}) \cdot \vec{V} = \lambda (\vec{u} \cdot \vec{V}) = \vec{u} \cdot (\lambda \vec{V})$ (5) $\vec{0} \cdot \vec{u} = 0$

Theorem. If θ is the angle between $\vec{u}$ and $\vec{v}$, then $\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot\cos\theta$.



Cor. Two vectors are perpendicular if and only if their dot product is 0

Remark. By the above figure, we see that $|\vec{v}|\cos\theta$ gives the length of the projection of $\vec{v}$ in $\vec{u}$ direction. so $\vec{u}\cdot\vec{v}$ is a measure of the "level of colinear" of the two vectors.

We define the scalar projection of $\vec{v}$ on $\vec{u}$ to be $\frac{\vec{u}}{|\vec{u}|}\cdot\vec{v}$

the vector projection of $\vec{v}$ on $\vec{u}$ to be $\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}\cdot\frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u}$

Example: $\vec{u} = \langle 1,1,2\rangle$, $\vec{v} = \langle -2,3,1\rangle$ Find the dot product of $\vec{u}$ and $\vec{v}$, and find the scalar and vector projection of $\vec{v}$ on $\vec{u}$

$\vec{u}\cdot\vec{v} = 1\times(-2) + 1\times 3 + 2\times 1 = 3$, $|\vec{u}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$

The scalar projection of $\vec{v}$ on $\vec{u}$ is: $\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$

The vector projection of $\vec{v}$ on $\vec{u}$ is:

$$\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} = \frac{3}{(\sqrt{6})^2}\cdot\langle 1,1,2\rangle = \frac{1}{2}\cdot\langle 1,1,2\rangle = \langle \tfrac{1}{2},\tfrac{1}{2},1\rangle$$