SURFACE INTEGRALS OF VECTOR FIELDS

Oriented Surfaces:

If it is possible to choose a unit normal vector $\vec{n}$ at every point (x,y,z) on a surface S such that $\vec{n}$ varies continuously over S, then S is called an oriented surface, and the given choice of $\vec{n}$ provides S with an orientation.

There are two orientations for any orientable surface.

Sometimes, there're special choice of orientation.

If a surface S is the graph of $z=g(x,y)$, we can take the unit normal vector at each point to be

$$\vec{n} = \frac{1}{\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}} \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle$$

we call it the upward orientation.

More general, if S is a parametric surface $\vec{r}(u,v)$, then we usually take the unit normal vector

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|}$$

as the orientation.

Example. The unit sphere is $\vec{r}(\phi,\theta)=\langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle$

So the orientation induced by $\vec{r}$ is

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle = \frac{1}{a}\vec{r}(\phi,\theta)$$

This gives the outward unit vector.

When a surface is closed, we define the positive orientation to be the one for which the unit normal vectors point outward.

With the definition of orientation of surface, we can define the surface integral of $\vec{F}$ over $\vec{S}$:

If $\vec{F}$ is a continuous vector field defined on an oriented surface $\vec{S}$ with unit normal vector $\vec{n}$, then the surface integral of F over $\vec{S}$

is : $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$

This integral is also called the flux of $\vec{F}$ across S

If S is parameterized by $\vec{r}(u,v)$, then $\vec{n} = \dfrac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|}$

$$\text{So } \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}(\vec{r}(u,v))\cdot\frac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|}\cdot\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|dA$$

$$= \iint_S \vec{F}(\vec{r}(u,v))\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)dA$$

Example. Find the flux of $\vec{F}(x,y,z) = \langle z, y, x\rangle$ across the unit sphere $x^2+y^2+z^2=1$.

We know $\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$

$\vec{F}(\vec{r}(\phi,\theta)) = \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta\rangle$

$\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v} = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$.

$$\text{So } \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\phi,\theta))\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)dA$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\cos\phi\sin^2\phi\cos\theta + \sin\phi\sin\theta\sin^2\phi\sin\theta + \sin\phi\cos\theta\sin\phi\cos\phi\right)d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta \, d\phi\, d\theta$$

$$= 2\int_0^{\pi}\sin^2\phi\cos\phi\, d\phi \int_0^{2\pi}\cos\theta\, d\theta + \int_0^{\pi}\sin^3\phi\, d\phi \int_0^{2\pi}\sin^2\theta\, d\theta$$

$$= \frac{4\pi}{3}$$

Another case is when the surface is a graph of $z = g(x,y)$, then if $\vec{F} = \langle P, Q, R\rangle$, and we assumes the upward orientation

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle P, Q, R\rangle \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle dA$$

$$= \iint_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \, dA.$$

Applications.

Electric Flux $\iint_S \vec{E}\cdot d\vec{S}$.

Heat Flow $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = -K\nabla u$ is the head flow

K is the heat conductivity of the substance

$u(x,y,z)$ is the temperature function.

STOKES' THEOREM, DIVERGENCE THEOREM

Stoke's Theorem.

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a differentiable vector field with continuous derivatives on an open region in $\mathbb{R}^3$ that contains S. Then:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

The Divergence Theorem.

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a differentiable vector field with continuous derivatives on an open region that contains E. Then:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} F \, dV$$

Maxwell's Equations:

The Maxwell's Equations are a set of Partial Differential Equations.

① $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

② $\vec{\nabla} \cdot \vec{B} = 0$

③ $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

④ $\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right)$

We will try to translate them into integral equations:

① $\frac{1}{\varepsilon_0}\iiint_{\Omega} \rho = \iiint_{\Omega} \vec{\nabla}\cdot\vec{E} = \oiint_{\partial\Omega} \vec{E}\cdot d\vec{S}$ (Electric flux leaving a volume is proportional to the charge inside)

② $0 = \iiint_{\Omega} \vec{\nabla}\cdot\vec{B} = \oiint_{\partial\Omega} \vec{B}\cdot d\vec{S}$ (No magnetic monopoles)

② $-\iint_{\Sigma} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = \iint_{\Sigma} \vec{\nabla}\times\vec{E}\cdot d\vec{S} = \oint_{\partial\Sigma} \vec{E}\cdot d\vec{r}$ (voltage induced in closed loop is proportional to the rate of change of the magnetic flux that the loop encloses

$$\begin{gathered} -\iint_{\Sigma} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} \\ \| \\ -\frac{d}{dt}\iint_{\Sigma} \vec{B}\cdot d\vec{S} \end{gathered}$$

④ $\mu_0 \iint_{\Sigma} \vec{J}\cdot dS + \mu_0\varepsilon_0 \iint_{\Sigma} \frac{\partial \vec{E}}{\partial t}\cdot d\vec{S} = \iint_{\Sigma} \vec{\nabla}\times\vec{B} \doteq \oint_{\partial\Sigma} \vec{B}\cdot d\vec{r}$

(The magnetic field induced around a closed loop is proportional to the electric current plus displacement current that the loop encloses.