# PARAMETRIC SURFACES

A parametric surface $S$ is a function from $D \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$:

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

The functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are called parametric equations of $S$.

Example. If $S$ is the graph of $z = f(x,y)$, we can write it as

$$\vec{r}(u,v) = \langle u,\ v,\ f(u,v) \rangle$$

Example. Find a parametric representation for the cylinder $x^2+y^2=4$ $0 \le z \le 1$

We are looking for two parameters that can describe the locations of points on the cylinder. We realize that points on this cylinder are determined by the angle and z-coordinate.

So $\vec{r}(u,v) = \langle 2\cos u,\ 2\sin u,\ v \rangle$, $0 \le u \le 2\pi$, $0 \le u \le 1$

Example. Find a parametric representation of $x^2+y^2+z^2=R^2$.

The positions of points on the sphere centered at $O$ are determined by the two angles $\theta$ & $\phi$ in spherical coordinates

So $\vec{r}(\theta,\phi) = \langle R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi \rangle$

$0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

Given a surface parameterized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, we would like to find its tangent surface.

The strategy is to find two tangent vectors, and these two tangent vectors will span the tangent plane.

For a given $(u_0, v_0)$, we first fix $v_0$ and consider the curve

$\vec{\alpha}(u) = \vec{r}(u, v_0) = \langle x(u, v_0), y(u, v_0), z(u, v_0)\rangle$

This curve lies on the surface $\vec{r}(u,v)$

$\vec{\alpha}'(u_0) = \frac{\partial \vec{r}}{\partial u}(u_0, v_0) = \langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\rangle$ is the tangent vector of $\vec{\alpha}(u)$ at $u = u_0$, so it's also a tangent vector for $\vec{r}(u,v)$.

Similarly, we can define

$\vec{\beta}(v) = \vec{r}(u_0, v) = \langle x(u_0, v), y(u_0, v), z(u_0, v)\rangle$

and obtain another tangent vector

$\vec{\beta}'(v_0) = \frac{\partial \vec{r}}{\partial v}(u_0, v_0) = \langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0)\rangle$

If $\vec{\alpha}'(u_0)$ & $\vec{\beta}'(v_0)$ are not colinear, we say the surface $\vec{r}(u,v)$ is smooth. Then it's tangent plane is the one passing through $\vec{r}(u_0, v_0)$ and perpendicular to $\vec{\alpha}'(u_0) \times \vec{\beta}'(v_0)$

Example. Find the tangent plane to the surface $\vec{r}(u,v) = \langle u^2, v^2, u+2v\rangle$ at the point $(1, 1, 3)$.

$$\begin{cases} \frac{\partial \vec{r}}{\partial u} = \langle 2u, 0, 1\rangle \\ \frac{\partial \vec{r}}{\partial v} = \langle 0, 2v, 2\rangle \end{cases}$$

so a normal vector to the tangent plane is

$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle -2v, -4u, 4uv\rangle$

$\vec{r}(1,1) = \langle 1, 1, 3\rangle$, so the normal vector at $\vec{r}(1,1) = \langle -2, -4, 4\rangle$

the tangent plane at $(1,1,3)$ is hence:

$$-2(x-1) - 4(y-1) + 4(z-3) = 0$$

We are going to study the surface area of a parametric surface.

By similar argument on Riemann Sum of the Change of Variable Section, we obtain the following definition:

If $S$ is parameterized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ and $S$ is covered just once as $(u,v)$ ranges throughout the parameter domain $D$, then the surface area of $S$ is

$$A(S) = \iint_D \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA.$$

A special case of that is when $S$ is the graph of $z = f(x,y)$

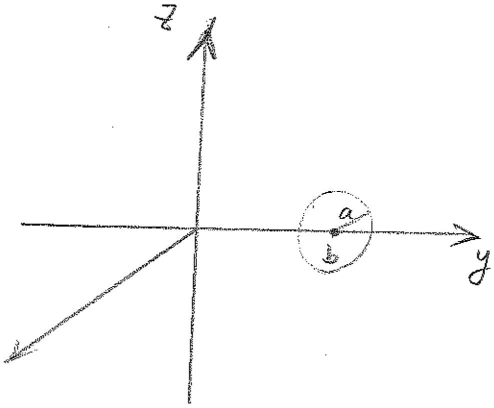
$\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$

Then $\begin{cases} \dfrac{d\vec{r}}{dx} = \langle 1, 0, \dfrac{\partial f}{\partial x}\rangle \\ \dfrac{\partial \vec{r}}{dy} = \langle 0, 1, \dfrac{\partial f}{\partial y}\rangle \end{cases}$ so $\dfrac{d\vec{r}}{dx} \times \dfrac{d\vec{r}}{dy} = \langle -\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1\rangle$.

we get the area of the surface is

$$A(S) = \iint_D \sqrt{1^2 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Example. Find the surface area of the torus obtained from rotating the circle in yz-plan $(y-b)^2+z^2=a^2$ $(0<a<b)$ around z-axis.




The equation of the circle is $(y-b)^2+z^2=a^2$.

So it can be parameterized to be

$y = b + a\cos u,\quad z = \sin u. \qquad 0 \le u \le 2\pi.$

Then the equation for the torus is

$$\vec{r}(u,v) = \langle (b+a\cos u)\cos v,\ (b+a\cos u)\sin v,\ a\sin u \rangle,\quad 0\le u \le 2\pi,\ 0 \le v \le 2\pi$$

$$\frac{\partial \vec{r}}{\partial u} = \langle -a\sin u\cos v,\ -a\sin u \sin v,\ a\cos u \rangle.$$

$$\frac{\partial \vec{r}}{\partial v} = \langle -(b+a\cos u)\sin v,\ (b+a\cos u)\cos v,\ 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle -a(b+a\cos u)\cos u\cos v,\ -a(b+a\cos u)\cos u \sin v,\ -a(b+a\cos u)\sin u$$

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{a^2(b+a\cos u)^2\cos^2 u\cos^2 v + a^2(b+a\cos u)^2\cos^2 u \sin^2 v + a^2(b+a\cos u)^2 \sin^2 u}$$

$$= a(b+a\cos u)$$

So the area is

$$\iint_D a(b+\cos u)\,dA = \int_0^{2\pi}\int_0^{2\pi} ab + a\cos u \,du\,dv$$

$$= \int_0^{2\pi} 2\pi ab\,dv$$

$$= 4\pi^2 ab$$

# SURFACE INTEGRALS

We are going to define the surface integral of a function $f(x,y,z)$ on a surface $S$ given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

In the previous section, we see the area of a surface is

$$A(S) = \iint_D \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

Which follows from the observation $dS = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$.

By the same principle, we can define the surface integral to be:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

Example. A thin sphere of radius 1 made of some mixed material has density function $\rho(x,y,z) = x^2$. Compute the mass of the sphere.

The sphere can be parameterized by

$$\vec{r}(\theta, \phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle, \quad 0 \le \phi \le \pi, \ 0 \le \theta \le 2\pi$$

$$\text{Mass} = \iint_S \rho(x,y,z)\, dS = \int_0^{\pi}\int_0^{2\pi} (\sin\phi\cos\theta)^2 \cdot \left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{d\phi} \right| d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} (\sin\phi\cos\theta)^2 \cdot \sin\phi\, d\theta\, d\phi$$

$$= \frac{4}{3}\pi$$

A special case is when the surface is a graph of function $z = g(x,y)$.

In such a case, we know by previous discussion, if we parameterize $S$ by $\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$, then $\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$

So the integral is.

$$\iint_S f(x,y,z)\,dA = \iint_D f(\vec{r}(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dA.$$

Example. Evaluate the previous mass in another way:

We can decompose the sphere into two hemispheres:

$z=\sqrt{1-x^2-y^2}$ and $z=-\sqrt{1-x^2-y^2}$, call them $S_1$ and $S_2$.

then

$$\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \iint_{S_2} f(x,y,z)\,dS$$

$$= \iint_{x^2+y^2\le 1} x^2\sqrt{1+\left(\frac{\partial}{\partial x}\sqrt{1-x^2-y^2}\right)^2+\left(\frac{\partial}{\partial y}\sqrt{1-x^2-y^2}\right)^2}\,dA$$

$$+ \iint_{x^2+y^2\le 1} x^2\cdot\sqrt{1+\left(\frac{\partial}{\partial x}\left(-\sqrt{1-x^2-y^2}\right)\right)^2+\left(\frac{\partial}{\partial y}\left(-\sqrt{1-x^2-y^2}\right)\right)^2}\,dA$$

$$= \iint_{x^2+y^2\le 1} x^2\cdot\frac{1}{\sqrt{1-x^2-y^2}}\,dA + \iint_{x^2+y^2\le 1} x^2\cdot\frac{1}{\sqrt{1-x^2-y^2}}\,dA$$

$$= 2\iint_{x^2+y^2\le 1} \frac{x^2}{\sqrt{1-x^2-y^2}}\,dA$$

$$= 2\int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}}\cdot r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\cos^2\theta\,d\theta\cdot\int_0^1\frac{r^3}{\sqrt{1-r^2}}\,dr$$

$$= 2\cdot\pi\cdot\frac{2}{3}$$

$$= \frac{4}{3}\pi$$