GREEN'S THEOREM

Consider a region D on the plane whose boundary is a simple closed curve C. We say that C is positively oriented if C has the counterclockwise orientation. i.e. if walking along the orientation of C, the region D is always to the left of C.

Green's Theorem.

Let C be a positively oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ and D is the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region containing D, then: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

In general, this theorem is not easy to prove. But if the region is of Type I and Type II, we can give a proof here.

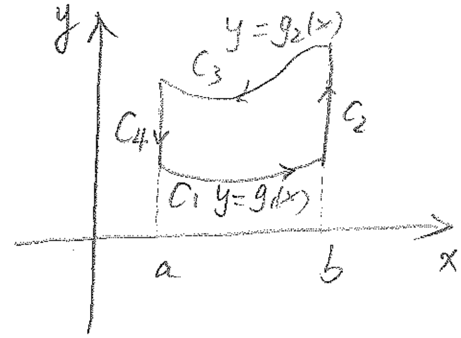
The strategy of the proof is to show:

$$\begin{cases} \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA & \cdots ① \\ \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA & \cdots ② \end{cases}$$

Since we assume D is of both Type I and Type II, we can assume $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ (Type I)

So $\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx$

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$$



$$= \int_{C_1} P\,dx + \int_{C_3} P\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx + \int_b^a P(x, g_2(x))\,dx$$

$$= \int_a^b P(x, g_1(x)) - P(x, g_2(x))\,dx$$

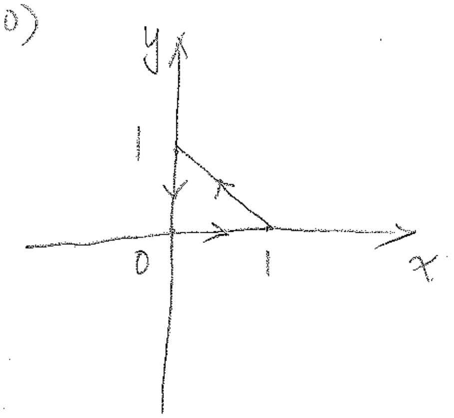
So we see $\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$

Similarly, we can use that D is a region of Type II to obtain

$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$, so we obtain the theorem.

The Green's Theorem tells us that we can evaluate the line integral of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ along a curve C by calculating the double integral $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$ instead.

Example: Evaluate $\int_C x^4\,dx + xy\,dy$, C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$



$$\int_C x^4\,dx + xy\,dy = \iint_D \frac{\partial(xy)}{\partial x} - \frac{\partial(x^4)}{\partial y}\,dA$$

$$= \int_0^1 \int_0^{1-x} y - 0\,dy\,dx$$

$$= \frac{1}{6}$$

Notation: When C is a positively oriented closed curve, the line integral of a vector field $\vec{F}$ along C is written as:

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P(x,y)\,dx + Q(x,y)\,dy$$

Example. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle: $x^2+y^2=9$

$$\oint_C (3y-e^{\sin x})\,dx + (7x+\sqrt{y^4+1})\,dy = \iint_{x^2+y^2\le 9} \frac{\partial(7x+\sqrt{y^4+1})}{\partial x} - \frac{\partial(3y-e^{\sin x})}{\partial y}\,dA$$

$$= \iint_{x^2+y^2\le 9} 7-3\,dA.$$

$$= \iint_{x^2+y^2\le 9} 4\,dA$$

$$= 4\iint_{x^2+y^2\le 9} 1\,dA$$

$$= 4\cdot\pi\cdot 3^2$$

$$= 36\pi.$$

Theorem. The area of the region D enclosed by a simple closed curve C is $A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$

Proof. We know that $A = \iint_D 1\,dA$

$$A = \iint_D 1\,dA = \iint_D \frac{\partial x}{\partial x} - \frac{\partial 0}{\partial y}\,dA = \oint_C 0\cdot dx + x\,dy = \oint_C x\,dy$$

$$= \iint_D \frac{\partial 0}{\partial x} - \frac{\partial(-y)}{\partial y}\,dA = \oint_C -y\,dx + 0\,dy = -\oint_C y\,dx$$

$$= \iint_D \frac{\partial(\frac{x}{2})}{\partial x} - \frac{\partial(-\frac{y}{2})}{\partial y}\,dA = \oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Example. Find the area inside an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We can parameterize the ellipse by $\vec{r}(t)=\langle a\cos t, b\sin t\rangle$.
$0\le t\le 2\pi$.

So $A=\frac{1}{2}\oint_C x\,dy-y\,dx=\frac{1}{2}\int_0^{2\pi} a\cos t\, d(b\sin t)-b\sin t\, d(a\cos t)$

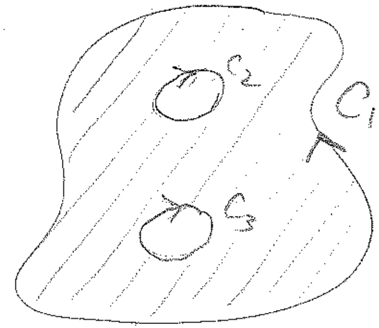
$$=\frac{1}{2}\int_0^{2\pi} ab\cos^2 t+ab\sin^2 t\, dt$$

$$=\frac{1}{2}\int_0^{2\pi} ab\, dt$$

$$=\pi ab$$

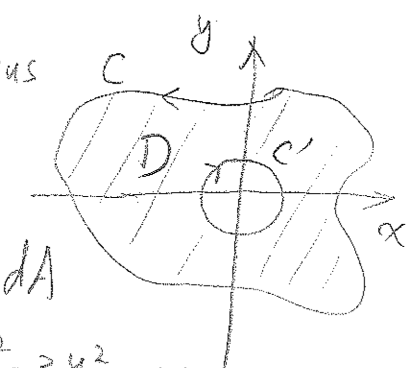
We can extend Green's Theorem to regions with holes.

If D is a region with n holes, then its boundary is a disjoint union of $n+1$ curves. We orient each C_i by requiring that walking along C_i, the region is to the left.



Example. If $\vec{F}(x,y)=\langle\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\rangle$, prove that $\int_C \vec{F}\cdot d\vec{r}=2\pi$ for every positively oriented simple closed path that contains $(0,0)$

We take a small circle C' centered at $(0,0)$ with radius r_0 so that C' doesn't intersect C.



Then $\oint_C \vec{F}(x,y)\cdot d\vec{r}-\oint_{C'}\vec{F}(x,y)\cdot d\vec{r}=\iint_D \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)dA$

$$=\iint_D \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}+\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\,dA$$

$$=\iint_D 0\, dA=0.$$

CURL AND DIVERGENCE

Consider a vector field $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ in $\mathbb{R}^3$ such that P, Q, R are differentiable, then we define:

The curl of $\vec{F}$: $\operatorname{curl} F = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$

Formally, we define $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ the vector differential operator.

Then we can check that

$$\operatorname{Curl} F = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial k} \\ P & Q & R \end{vmatrix}$$

Example. $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$; then

$$\operatorname{Curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz), \frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(-y^2), \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \right\rangle$$

$$= \langle -2y - xy, x, yz \rangle$$

Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{Curl}(\nabla f) = \vec{0}$$

Proof. $\operatorname{Curl}(f) = \vec{\nabla} \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$

$$= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle$$

$$= \vec{0}$$

There is a useful application of the theorem:

Example. Show that $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$ is not conservative

$\vec{\nabla} \times \vec{F} = \langle -2y - xy, x, yz \rangle \neq \vec{0}$, so $\vec{F} \neq \nabla f$ for any f.

As to the converse, we have the following theorem:

Theorem. If $\vec{F}$ is a vector field on whole $\mathbb{R}^3$, whose component functions have continuous partial derivatives and $\text{Curl } \vec{F} = 0$, then $\vec{F}$ is a conservative vector field.

Example. Show that $\vec{F}(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ is a conservative vector field.

$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = \vec{0}$, so it's conservative.

If $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field on $\mathbb{R}^3$ and P, Q, R are differentiable, then define the divergence of $\vec{F}$ to be:

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

We can write $\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$.

Example. $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$.

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

Theorem. If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second-order partial derivatives, then:

$$\operatorname{div} \operatorname{curl} \vec{F} = 0$$

Pf. $\operatorname{div} \operatorname{curl} \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$

Example. Show that $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$ can't be written as $\operatorname{curl} \vec{G}$ for any $\vec{G}$.

$\operatorname{div} \vec{F} = z + xz \neq 0$, so we can't write $\vec{F} = \operatorname{curl} \vec{G}$.

We define $\vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ to be the Laplace operator.

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We can also let $\vec{\nabla}^2$ acts on a vector field $\vec{F} = \langle P, Q, R \rangle$ by

$$\vec{\nabla}^2 \vec{F} = \langle \vec{\nabla}^2 P, \vec{\nabla}^2 Q, \vec{\nabla}^2 R \rangle$$

With the help of the curl, we can find a new expression for the Green's Theorem.

We regard a vector field $\vec{F} = \langle P, Q \rangle$ in $\mathbb{R}^2$ as a vector field $\vec{F} = \langle P, Q, 0 \rangle$ in $\mathbb{R}^3$. then

$$\operatorname{Curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$$

So Green's Theorem becomes: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot \vec{k} \, dA$

With the help of divergence, we can also express Green's Theorem in another way:

We parameterize the simple closed curve C by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

We let $\vec{n}(t) = \frac{1}{|\vec{r}'(t)|} \langle y'(t), -x'(t) \rangle$, then $\vec{n}$ is the unit normal vector of C at $\vec{r}(t)$ pointing outwards.

We let $s = s(t) = \int_a^t |\vec{r}'(u)| du$, the arc length

Then:
$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t)\, |\vec{r}'(t)|\, dt$$
$$= \int_a^b \frac{P(\vec{r}(t)) \cdot y'(t) - Q(\vec{r}(t)) \cdot x'(t)}{|\vec{r}'(t)|} \cdot |\vec{r}'(t)|\, dt$$
$$= \oint_C P\, dy - Q\, dx$$
$$= \oint_C -Q\, dx + P\, dy$$
$$= \iint_D \frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}\, dA.$$
$$= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA$$
$$= \iint_D \operatorname{div} \vec{F}\, dA$$

So we can write the Green's theorem as

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$$