THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Theorem. Let C be a smooth curve given by the vector function $\vec{r}(t)$ $a \le r \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C, then:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof.

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \, dt$$

$$= \int_a^b \frac{d f(\vec{r}(t))}{dt} \, dt$$

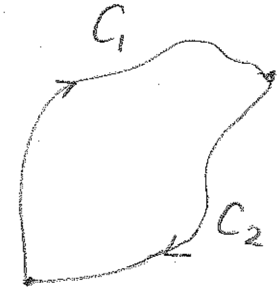
$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

A remark is that the theorem is still true if we replace smooth curves by piecewise-smooth curves.

The Fundamental Theorem for Line Integrals implies that the line integral of a conservative vector field only depends on the initial and terminal positions of the curve, i.e. if C_1 and C_2 are two curves such that $\vec{r}_1(a) = \vec{r}_2(a)$, $\vec{r}_1(b) = \vec{r}_2(b)$, then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$

We say $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 that have the same initial and terminal points.

Theorem. $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}\cdot d\vec{r}=0$ for every closed path C in D.



Proof. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D.

then any closed path C can be decomposed into two parts C_1 and C_2

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} = 0$$

so $\int_C \vec{F}\cdot d\vec{r}=0$ for every closed path C in. D

If $\int_C \vec{F}\cdot d\vec{r}=0$ for every closed path C in D.

then for any two curves C_1 and C_2 with same initial and terminal point, then C_1-C_2 is a closed path. so

$$\int_{C_1-C_2} \vec{F}\cdot d\vec{r}=0, \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = 0 \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

Theorem. $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

Proof. We need to construct f for a given $\vec{F}$

Since $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D,

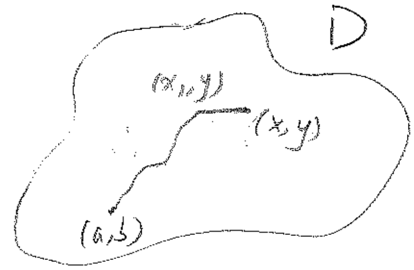
once we fix $(a,b)\in D$, we can define for any $(x,y)\in D$:

$$f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$$

We need to show $\nabla f(x,y) = \vec{F}(x,y)$.

Assume $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$.

then we need to show $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$.



For a given $(x,y) \in D$, we choose a path from (a,b) to (x,y) such that near (x,y) it's a horizontal segment approaching (x,y) from left, say the segment is (x_1,y) — (x,y).

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left[\int_{(a,b)}^{(x_1,y)} \vec{F}(x,y)\cdot d\vec{r} + \int_{(x_1,y)}^{(x,y)} \vec{F}(x,y)\cdot d\vec{r}\right]$$

$$= \frac{\partial}{\partial x}\int_{(x_1,y)}^{(x,y)} \vec{F}(x,y)\cdot d\vec{r}$$

$$= \frac{\partial}{\partial x}\int_{(x_1,y)}^{(x,y)} P\,dx + Q\,dy$$

parameterize the segment by $\alpha(t) = \langle t, y \rangle$, $x_1 \le t \le x$

$$= \frac{\partial}{\partial x}\int_{x_1}^{x} P(t,y)\,dt$$

$$= P(x,y)$$

Similarly, we can show that $\frac{\partial}{\partial y} f(x,y) = Q(x,y)$

Next, we would like to consider if there is a more "algebraic" method to tell if a vector field is conservative or not.

A necessary condition is given by the following theorem:

Theorem. If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is conservative, where P & Q have continuous partial derivatives on D, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

But we would also like to see if there is a sufficient condition.

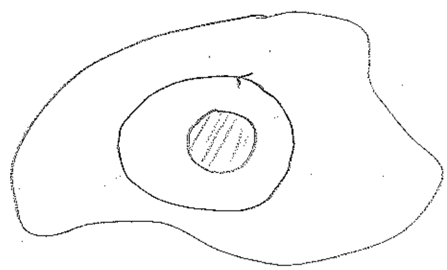
It turns out such condition holds when we consider a special kind of domain.

We define a simple curve to be a curve that does not intersect itself anywhere between its endpoints. (The only possible intersection is $\vec{r}(a) = \vec{r}(b)$, which makes $\vec{r}(t)$, $a \le t \le b$ into a simple closed curve)

Next, we define a simply-connected region in the plane to be a connected region D such that every simple closed curve in D encloses only points that are in D.

Example. • If there is a "hole" on a region, then the region cannot be simply-connected.

• If the region is not connected, it's also not simply-connected



We will introduce the following theorem:

Let $\vec{F} = \langle P, Q \rangle$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D, then $\vec{F}$ is conservative.

Example. Determine whether or not $\vec{F}(x,y) = \langle x-y,\ x-2 \rangle$ is conservative

$$\frac{\partial}{\partial y}(x-y) = -1, \quad \frac{\partial}{\partial x}(x-2) = 1, \quad \text{so } \vec{F}(x,y) \text{ is not conservative}$$

Example. Determine whether or not $\vec{F}(x,y) = \langle 3+2xy,\ x^2-3y^2 \rangle$ is conservative.

$$\frac{\partial}{\partial y}(3+2xy) = 2x, \quad \frac{\partial}{\partial x}(x^2-3y^2) = 2x$$

and the region $\mathbb{R}^2$ is open and simply-connected.

So $\vec{F}(x,y)$ is conservative.

Example. If $\vec{F}(x,y) = \langle 3+2xy,\ x^2-3y^2 \rangle$ Find a potential function.

Method I:

If $f(x,y)$ is a potential function, then

$$\nabla f(x,y) = \vec{F}(x,y)$$

so
$$\begin{cases} \dfrac{\partial f}{\partial x} = 3+2xy & \cdots ① \\ \dfrac{\partial f}{\partial y} = x^2-3y^2 & \cdots ② \end{cases}$$

We may first integrate ① w.r.t. x : $f(x,y) = 3x + x^2y + g(y)$

then $\frac{\partial f}{\partial y} = x^2 + g'(y)$ so by ②, $g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + K$

So $f(x,y) = 3x + x^2y - y^3 + K$

We may take $K=0$ so $f(x,y) = 3x + x^2y - y^3$

Method II:

$$f(x_0,y_0) = \int_{(0,0)}^{(x,y)} (3+2xy)\,dx + (x^2-3y^2)\,dy$$

$$= \int_0^1 (3+2t^2x_0y_0)\,x_0\,dt + (t^2x_0^2 - 3t^2y_0^2)\,y_0\,dt$$

$$= \int_0^1 3x_0 + (3x_0^2y_0 - 3y_0^3)\,t^2\,dt$$

$$= 3x_0 + x_0^2y - y_0^3$$

Conservation of Energy :

$\vec{F}(\vec{r})$ is a force field, that is, at a point $\vec{r} = \langle x, y, z \rangle$, the force is $\vec{F}(x,y,z)$

Newton's Second Law of Motion is $\vec{F}(\vec{r}(t)) = m\vec{r}''(t)$

so if we consider the work done by the force field on an object which is moving along a curve C due to the force is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$$= \int_a^b m\,\vec{r}''(t) \cdot \vec{r}'(t)\,dt$$

$$= \int_a^b m \cdot \frac{(\vec{r}'(t)\vec{r}'(t))'}{2}\,dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |\vec{r}'(t)|^2\,dt$$

$$= \frac{m}{2} \left(|\vec{r}'(b)|^2 - |\vec{r}'(a)|^2 \right)$$

We define the kinetic energy of an object to be $K(\vec{r}(t)) = \frac{1}{2} m \cdot |\vec{r}'(t)|^2$

So $W = K(\vec{r}(b)) - K(\vec{r}(a))$.

We define the potential energy of an object at $\vec{r}$ in a conservative force field $\vec{F} = \nabla f$ to be $P(\vec{r}) = -f(\vec{r})$.

So $W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = -\int_C \nabla P(\vec{r})\,d\vec{r} = P(\vec{r}(a)) - P(\vec{r}(b))$

So $K(\vec{r}(b)) - K(\vec{r}(a)) = P(\vec{r}(a)) - P(\vec{r}(b))$

i.e. $K(\vec{r}(a)) + P(\vec{r}(a)) = K(\vec{r}(b)) + P(\vec{r}(b))$

This means if an object moves under the influence of a conservative force field, then $K + P$, its energy is a constant