VECTOR FIELD

Let D be a set in $\mathbb{R}^n$. A vector field on $\mathbb{R}^n$ is a function $\vec{F}$ that assigns to each point $(x_1, \dots, x_n)$ in D an n-dimensional vector $\vec{F}(x_1, \dots, x_n)$

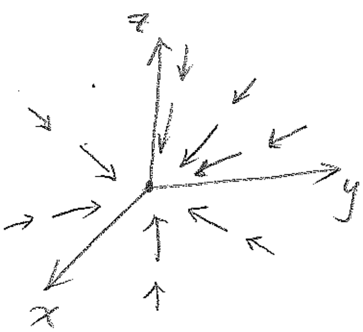
In particular, we are interested in 2 and 3 dimensional vector fields.

A two dimensional vector field can be written as $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$.
Similarly, a three dimensional vector field can be written as

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

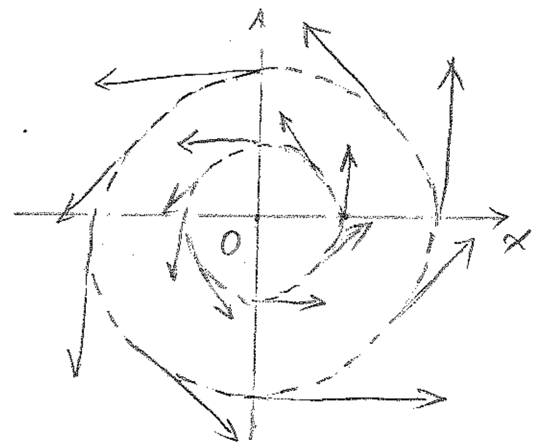
Example. Newton's Law of Gravitation

$$\vec{F}(x,y,z) = -\frac{mMG}{|\langle x,y,z \rangle|^3} \cdot \langle x,y,z \rangle.$$



Example. On $\mathbb{R}^2$, we can define the vector field

$$\vec{F}(x,y) = \langle -y, x \rangle.$$



There is a special type of vector field we would like to define now.

If f is a function of n variables, we define the gradient vector field of f to be $\vec{F}(x_1, \dots, x_n) = \nabla f(x_1, \dots, x_n) = \langle \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) \rangle$

We call a vector field a conservative vector field if it is the gradient of some scalar function f. i.e. $\vec{F}(x_1, \dots, x_n) = \nabla F(x_1, \dots, x_n)$ and f is called the potential function for $\vec{F}(x_1, \dots, x_n)$

Example. The gravitional field is a conservative vector field with potential function $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$

Example. $\vec{F}(x,y) = \langle -y, x \rangle$ is NOT a conservative vector field, since suppose $\vec{F}(x,y) = \nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ for some function $f(x,y)$

then
$$\begin{cases} \dfrac{\partial f}{\partial x} = -y \\[2mm] \dfrac{\partial f}{\partial y} = x \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial y \partial x} = -1 \\[2mm] \dfrac{\partial^2 f}{\partial x \partial y} = 1 \end{cases}$$
contradiction.

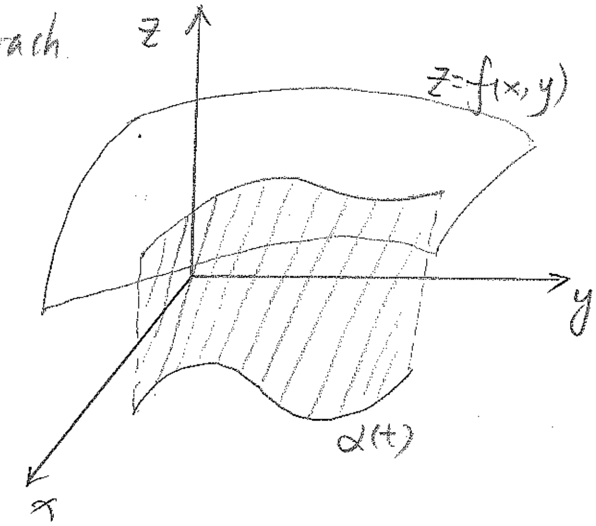
LINE INTEGRALS

Recall the integral $\int_a^b f(x)dx$ represents the area bounded by the graph of the function f and the x axis between a and b.

Now we would like to generalize it to the following case:

Consider a curve $C: \alpha(t) = (x(t), y(t))$ on the xy-plane and a two variable function $f(x,y)$. We would like to study the area of the surface above the curve $\alpha(t)$ and below the graph of $f(x,y)$.

The natural trial is the Riemann Sum Approach



Let $\alpha(t)$ be defined on $a \le t \le b$

and we divide the time interval $[a,b]$

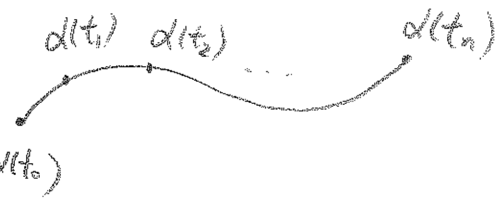
into many small pieces:

$a = t_0 < t_1 < t_2 < \cdots < t_n = b$

and choose a representative $t_i^* \in [t_{i-1}, t_i]$

Then the curve is divided into many small pieces by the points $\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_n)$. The length of the arc between $\alpha(t_{i-1})$ and $\alpha(t_i)$ can be estimated by $\sqrt{x'(t_i^*)^2 + y'(t_i^*)^2}\,\Delta t_i$

So consider the Riemann Sum:



$$\sum_{i=0}^{n} f(\alpha(t_i^*))\sqrt{x'(t_i^*)^2 + y'(t_i^*)^2}\,\Delta t_i$$

When we take the limit of the Riemann Sum as $\max \Delta t_i \to 0$

the Riemann Sum converges to the area of the surface we want.

So define $\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ to be the line integral of $f(x,y)$ along the curve C

A remark is that the value $\int_C f(x,y)\,ds$ doesn't depend on the choice of parametrization of the curve C, if C is traversed exactly once at each point.

The ordinary Riemann Sum is a special case of the line integral, where we take C to be the interval $[a,b]$ on the x-axis.

i.e. $\int_a^b f(x,0)\,dx = \int_a^b f(x,y)\,ds$ where C is $\alpha(x) = (x,0)$.

Example. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper-half of the unit circle $x^2+y^2=1$.

C can be parameterized to be $\alpha(t) = (\cos t, \sin t)$, $0 \le t \le \pi$

so
$$\int_0^\pi (2+x^2y)\,ds = \int_0^\pi (2+\cos^2 t \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt$$
$$= \int_0^\pi 2 + \cos^2 t \sin t\,dt$$
$$= 2t - \frac{\cos^3 t}{3}\Big|_0^\pi$$
$$= 2\pi + \frac{2}{3}$$

Example. Evaluate $\int_C 2x\,ds$, where C consists of arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

C_1 : $\alpha(t) = (t, t^2)$, $0 \le t \le 1$.

C_2 : $\beta(t) = (1, t)$, $1 \le t \le 2$.

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \int_0^1 2t\sqrt{1^2+(2t)^2}\,dt + \int_1^2 2\sqrt{0^2+1^2}\,dt$$
$$= \int_0^1 2t\sqrt{4t^2+1}\,dt + \int_1^2 2\,dt$$
$$= \frac{5\sqrt{5}-1}{6} + 2$$

We can also consider the projection of the surface above a curve C and below graph of $z=f(x,y)$ onto the xz-plane and yz-plane.

The corresponding areas of the projections on the xz-plane and yz-plane are given by $\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$ and

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

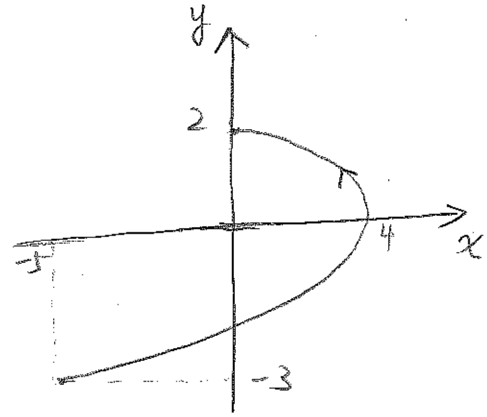
They're called the line integrals of f along C with respect to x and y.

Sometimes we need to integrate $P(x,y)$ along C with respect to x, and also integrate $Q(x,y)$ along C with respect to y at the same time. In this case, we usually write

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

Example. Evaluate $\int_C y^2\,dx + x\,dy$, where C is the arc of the parabola $x=4-y^2$ from $(-5,-3)$ to $(0,2)$

parameterize C as $\alpha(t)=(4-t^2,\ t)$, $-3\le t\le 2$.



$$\int_C y^2\,dx + x\,dy = \int_{-3}^{2} t^2(4-t^2)'\,dt + \int_{-3}^{2}(4-t^2)\,t'\,dt$$

$$= \int_{-3}^{2} -2t^3\,dt + \int_{-3}^{2} 4-t^2\,dt$$

$$= \frac{245}{6}$$

Exercise. Evaluate the above line integral by changing the path to the line segment from $(-5,-3)$ to $(0,2)$

Hint: parameterize by $\beta(t) = ((1-t)\cdot(-5)+t\cdot 0,\ (1-t)\cdot(-3)+t\cdot 2)$

$$= (5t-5,\ 5t-3), \quad 0\le t\le 1$$

Theorem. $\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$, $\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$

$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$

Three Dimensional Case:

The line integral of a function $f(x,y,z)$ along a curve C parameterized by $\alpha(t) = (x(t), y(t), z(t))$ is given by:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

We can also write the curve C as $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$.

then $\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$.

When $f(x,y,z) \equiv 1$, $\int_C 1\,ds = \int_a^b |\vec{r}\,'(t)|\,dt$ = length of the curve.

Also similarly, we can define $\int_C f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$.

and $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$

All the computations are similar to the case of two dimensions.

Line Integrals of Vector Fields:

Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is defined by $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$.

Example. Find the work done by the force field $\vec{F}(x,y)=\langle x^2,-xy\rangle$ in moving a particle along the quarter-circle $\vec{r}(t)=\langle \cos t, \sin t\rangle$, $0\le t\le \frac{\pi}{2}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \langle \cos^2 t, -\cos t\sin t\rangle \cdot \langle -\sin t, \cos t\rangle dt$$

$$= \int_0^{\frac{\pi}{2}} -2\cos^2 t \sin t \, dt$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t \, d\cos t$$

$$= \frac{\cos^3 t}{3}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{3}$$

If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$, $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$

then $$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(P(\vec{r}(t))\cdot x'(t) + Q(\vec{r}(t))\cdot y'(t) + R(\vec{r}(t))\cdot z'(t)\right) dt$$

$$= \int_a^b P(\vec{r}(t))\cdot x'(t)dt + Q(\vec{r}(t))y'(t)dt + R(\vec{r}(t))z'(t)dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

Actually, we can regard $d\vec{r} = \langle dx, dy, dz\rangle$ to understand this equality.