1. Differentiate the functions:

(i). $f(x) = x^3 - \frac{1}{\sqrt[4]{x^3}}$

Solution:

$f(x) = x^3 - x^{-\frac{3}{4}}$

$f'(x) = 3x^2 + \frac{3}{4}x^{-\frac{7}{4}}$

(ii). $f(x) = \frac{\sin x}{2} + \frac{2}{x} + 3$

Solution:

$f'(x) = \frac{\cos x}{2} - \frac{2}{x^2}$

(iii). $f(x) = x^3 \cos x$

Solution:

$f'(x) = (x^3)' \cos x + x^3 (\cos x)' = 3x^2 \cos x - x^3 \sin x$

(iv). $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

Solution:

$f'(x) = \frac{(\sqrt{x}-1)'(\sqrt{x}+1)-(\sqrt{x}-1)(\sqrt{x}+1)'}{(\sqrt{x}+1)^2} = \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}+1)-(\sqrt{x}-1)\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

(v). $f(x) = x^2 \sin x \tan x$

Solution:

$$\begin{aligned} f'(x) &= (x^2)' \sin x \tan x + x^2 (\sin x)' \tan x + x^2 \sin x (\tan x)' \\ &= 2x \sin x \tan x + x^2 \cos x \tan x + \frac{x^2 \sin x}{\cos^2 x} \end{aligned}$$

(vi). $f(x) = \frac{\cos x}{1-\sin x}$

Solution:

$$
\begin{aligned}
f'(x) &= \frac{(\cos x)'(1-\sin x) - (\cos x)(1-\sin x)'}{(1-\sin x)^2} \\
&= \frac{-\sin x(1-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\
&= \frac{-\sin x + 1}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}
\end{aligned}
$$

2. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

   Solution:

   $y' = 18x^2 + 5$. If $y' = 4$, then $18x^2 + 5 = 4$, so $x^2 = -\frac{1}{18} < 0$, which has no solution. We conclude the curve has no tangent line with slope 4.

3. Find equations of the tangent line to the curve $y = \frac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$.

   Solution:

   The line $x - 2y = 2$ has slope $\frac{1}{2}$.

   $y' = (\frac{x-1}{x+1})' = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

   If $y' = \frac{1}{2}$, we get $\frac{2}{(x+1)^2} = \frac{1}{2}$, i.e. $(x+1)^{=}4$, so $x = 1$ or $x = -3$.

   When $x = 1$, $y = \frac{1-1}{1+1} = 0$, so the tangent line is $y = \frac{1}{2}(x-1)$.

   When $x = -3$, $y = \frac{-3-1}{-3+1} = 2$, so the tangent line is $y - 2 = \frac{1}{2}(x+3)$

4. The cost function of producing x units of goods is given by

   $$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$$

   (a). Find the marginal cost function

   Solution:

$C'(x) = 12 - 0.2x + 0.0015x^2$

(b). Find $C'(200)$. What does it predict?

Solution:

$C'(200) = 12 - 0.2 \times 200 + 0.0015 \times 200^2 = 32$. It predicts approximately the increase in cost if the production increases by 1 unit from 200 units.

(c). Estimate the extra cost when the production is increased from 200 to 202.

Solution:

$C(202) - C(200) \approx C'(200) \times 2 = 32 \times 2 = 64$

5. A, B are constants. $y = A \sin x + B \cos x$ satisfies the equation

$$y'' + y' - 2y = 0$$

Determine the value of A and B.

Solution:

$y' = A \cos x - B \sin x$, $y'' = -A \sin x - B \cos x$

$y'' + y' - 2y = 0$, so:

$$\begin{aligned} -A \sin x - B \cos x + A \cos x - B \sin x - 2(A \sin x + B \cos x) &= 0 \\ (-3A - B) \sin x + (A - 3B) \cos x &= 0 \end{aligned}$$

We need $-3A - B = 0$ and $A - 3B = 0$, which implies $A = B = 0$.