DERIVATIVE FUNCTION

Given a function \( f(x) \), we can define the derivative function \( f'(x) \) by assigning the value of the derivative \( f(x) \) to \( x \), if the derivative exists at \( x \). So the domain of \( f'(x) \) is the set of numbers \( a \) at which the limit

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

exists.

Example. If \( f(x) = x^3 - x \), find a formula for \( f'(x) \).

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} - \frac{3x^2 - x}{h} \\
&= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - x^3) - (x^3 - x)}{h} \\
&= \lim_{h \to 0} \frac{3x^3h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h \to 0} \left( 3x^2 + 3xh + h^2 - 1 \right) \\
&= 3x^2 - 1
\end{align*}
\]

Example. If \( f(x) = \sqrt{x} \), find the derivative \( f'(x) \) and state its domain.

\[
\begin{align*}
f(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h \left( \sqrt{x+h} + \sqrt{x} \right)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad \text{exists when } x > 0.
\end{align*}
\]

so domain of \( f(x) \) is \((0, +\infty)\).
Example. Use the graph of $f$ to sketch the graph of $f'$. 

We used the relation that $f'(a)=0$ means the tangent line of $f(x)$ at $(a, f(a))$ has zero slope, i.e., its horizontal.

Notation. If we write $y=f(x)$ for a given function, then its derivative function can be denoted in the following ways:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

And we denote the value of $f'(x)$ at a number $a$ by

$$f'(a) = \frac{df}{dx}(a) = \left. \frac{dy}{dx} \right|_{x=a}$$

Definition. A function $f$ is differentiable at $a$ if $f'(a)$ exists. It's a differentiable function on an open interval $(a, b)$ (or $(a, +\infty)$, $(-\infty, a)$, $(-\infty, +\infty)$) if it's differentiable at every number in the interval.
Example. We have shown in the previous section that $f(x) = |x|^{1}$ has no derivative at $x=0$, so it's not differentiable at $x=0$.

For any $x > 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} 1 = 1$$

so $f(x)$ is differentiable on $(0, +\infty)$.

Similarly, for any $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{-(x+h)-(x)}{h} = \lim_{h \to 0} (-1) = -1$$

so $f(x)$ is differentiable on $(-\infty, 0)$.

Theorem. If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

The above example indicates the converse of the theorem is not true: $f(x) = |x|^{1}$ is continuous at $x = 0$, but it's not differentiable at $x = 0$.

There are several possible reasons for a function to fail to be differentiable at $x = a$:

(i) The graph has a "corner" or "kink" at $x = a$.
   This is usually caused by $\lim_{x \to a^-} \frac{f(a+h)-f(a)}{h} \neq \lim_{x \to a^+} \frac{f(a+h)-f(a)}{h}$.

(ii) The function is not continuous at $x = a$.

(iii) The function has vertical tangent line at $x = a$.
   i.e. $\lim_{x \to a} |f(x)| = \infty$.
HIGHER ORDER DERIVATIVES

In many cases, we can take the derivative of the derivative \( f'(x) \) of a function \( f(x) \). We call this "derivative of derivative" to be the second derivative, and denoted as

\[ f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx} \left( \frac{df}{dx} \right) \]

Example. \( S = S(t) \) is the displacement function of an object on the real line. Then our previous discussion shows that

\[ V(t) = S'(t) = \frac{dS}{dt} \]

is the velocity function of the object. Now we would like to study how fast the velocity is changing at each time, so we take the derivative of \( V(t) \) to measure its rate of change:

\[ A(t) = V'(t) = S''(t) = \frac{d^2S}{dt^2} \]

We call \( A(t) \) the acceleration of the object at time \( t \).

Similar to second derivative, if we take the derivative of \( f(x) \) \( n \) times, we obtain the \( n \)-th derivative, denoted as

\[ y^{(n)} = f^{(n)}(x) = \frac{d^n f}{dx^n} \]

Example. Compute \( f^{(n)}(x) \) for \( f(x) = x^3 \).

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \]

\[ = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \]

\[ = \lim_{h \to 0} \frac{(3x^2 + 3xh + h^2)}{h} \]

\[ = 3x^2 \]
\[
\begin{align*}
f''(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} \\
&= \lim_{x \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h} \\
&= \lim_{x \to 0} (6x + 3h) \\
&= 6x
\end{align*}
\]

\[
\begin{align*}
f'''(x) &= \lim_{h \to 0} \frac{f(x+h) - 6x}{h} = \lim_{h \to 0} \frac{6x + 6h - 6x}{h} \\
&= \lim_{h \to 0} 6 \\
&= 6
\end{align*}
\]

\[
\begin{align*}
f^{(4)}(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6 - 6}{h} = 0
\end{align*}
\]

If we continue, we'll see all the higher derivatives are 0.