DERIVATIVE FUNCTION

Given a function $f(x)$, we can define the derivative function $f'(x)$ by assigning the value of the derivative $f'(x)$ to x, if the derivative exists at x. So the domain of $f'(x)$ is the set of numbers a at which the limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists.

Example. If $f(x) = x^3 - x$, find a formula for $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3-(x+h)] - [x^3-x]}{h}$$

$$= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3-x-h) - (x^3-x)}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-h}{h}$$

$$= \lim_{h\to 0} (3x^2+3xh+h^2-1)$$

$$= 3x^2-1$$

Example. If $f(x) = \sqrt{x}$, find the derivative $f'(x)$ and state its domain.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

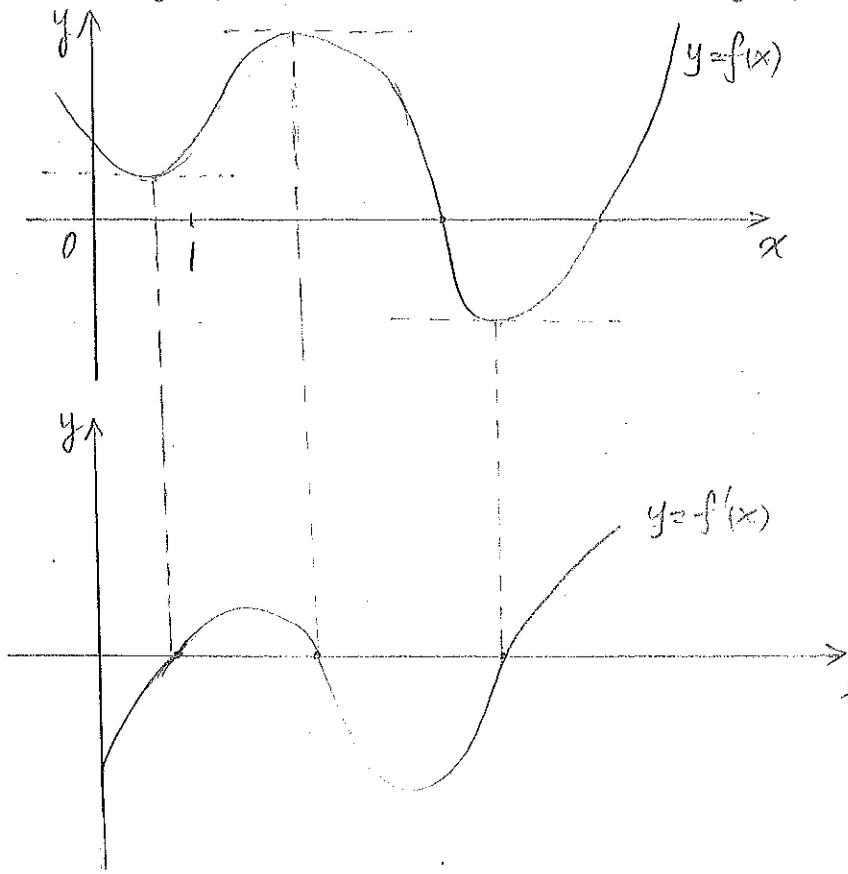
$$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$f'(x) = \frac{1}{2\sqrt{x}}$ exists when $x > 0$

so domain of $f'(x)$ is $(0, +\infty)$

Example. Use the graph of f to sketch the graph of f'.



We used the relation that $f'(a)=0$ means the tangent line of $f(x)$ at $(a, f(a))$ has zero slope, i.e, it's horizontal.

Notation. If we write $y=f(x)$ for a given function, then its derivative function can be denoted in the following ways:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

And we denote the value of $f'(x)$ at a number a by

$$f'(a) = \frac{df}{dx}(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

Definition. A function f is differentiable at a if $f'(a)$ exists. It's a differentiable function on an open interval (a, b) (or $(a, +\infty)$, $(-\infty, a)$, $(-\infty, +\infty)$) if it's differentiable at every number in the interval.

Example. We have shown in the previous section that $f(x)=|x|$ has no derivative at $x=0$, so it's not differentiable at $x=0$.

For any $x>0$.

$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}=\lim_{h\to 0}\frac{x+h-x}{h}=\lim_{h\to 1} 1 = 1.$$

So $f(x)$ is differentiable on $(0,+\infty)$

Similarly, for any $x<0$,

$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}=\lim_{h\to 1}(-1)=-1$$

so $f(x)$ is differentiable on $(-\infty,0)$

Theorem. If f is differentiable at a, then f is continuous at a.

The above example indicates the converse of the theorem is not true: $f(x)=|x|$ is continuous at $x=0$, but it's not differentiable at $x=0$

There are several possible reasons for a function to fail to be differentiable at $x=a$:

(i) The graph has a "corner" or "kink" at $x=a$

This is usually caused by $\lim_{x\to a^-}\frac{f(a+h)-f(a)}{h}\neq\lim_{x\to a^+}\frac{f(a+h)-f(a)}{h}$.

(ii) The function is not continuous at $x=a$.

(iii) The function has vertical tangent line at $x=a$.

i.e. $\lim_{x\to a}|f'(x)|=\infty$

HIGHER ORDER DERIVATIVES

In many cases, we can take the derivative of the derivative $f'(x)$ of a function $f(x)$. We call this "derivative of derivative" to be the second derivative, and denoted as

$$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

Example. $S = S(t)$ is the displacement function of an object on the real line. Then our previous discussion shows that

$v(t) = s'(t) = \frac{ds}{dt}$ is the velocity function of the object.

Now we would like to study how fast the velocity is changing at each time, so we take the derivative of $v(t)$ to measure its rate of change:

$$a(t) = v'(t) = s''(t) = \frac{d^2s}{dt^2}$$

We call $a(t)$ the acceleration of the object at time t.

Similar to second derivative, if we take the derivative of $f(x)$ n times, we obtain the n-th derivative, denoted as $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$.

Example. Compute $f^{(n)}(x)$ for $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= \lim_{h\to 0} (3x^2+3xh+h^2)$$

$$= 3x^2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= \lim_{x \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}$$

$$= \lim_{x \to 0} (6x + 3h)$$

$$= 6x$$

$$f'''(x) = \lim_{h \to 0} \frac{6(x+h) - 6x}{h} = \lim_{h \to 0} \frac{6x + 6h - 6x}{h}$$

$$= \lim_{h \to 0} 6$$

$$= 6$$

$$f^{(4)}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6 - 6}{h} = 0$$

if we continue, we'll see all the higher derivatives are 0.