DERIVATIVES AND RATES OF CHANGE

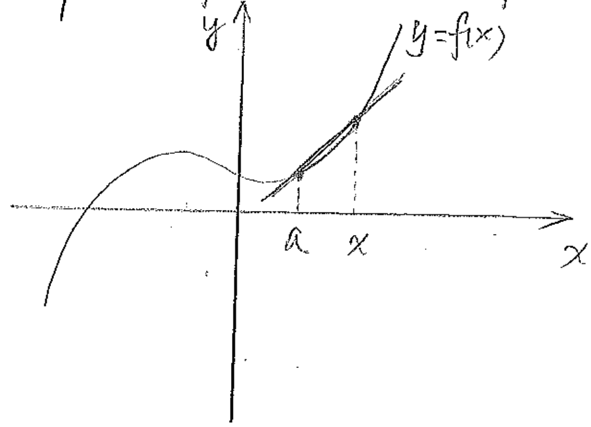
Definition: The tangent line to the curve $y=f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

provided the limit exists

We use the tangent line of $f(x)$ at $(a, f(a))$ to represent how steep the curve of $f(x)$ is at $(a, f(a))$.

The concept is formulated as follows:



for a given number x, consider the pair of points $(a, f(a))$ and $(x, f(x))$. There is a line passing through the two points, with slope to be

$$\frac{f(x)-f(a)}{x-a}$$

that is, the ratio of the change in y-coordinate over the change in x-coordinate.

The geometric observation is that if x approaches a, the slope of this line becomes better and better approximation to the curve $y=f(x)$ near $x=a$. So taking the limit as $x\to a$, we obtain a number which reflects the steepness of $y=f(x)$ at $x=a$.

Another way of formulation is by a change of variable.
let $h = x - a$, then $x \to a$ if and only if $h \to 0$.

So $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example. Let $f(x) = \frac{1}{x}$. At $x = 1$, the tangent line has slope

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{-\frac{h}{1+h}}{h}$$

$$= \lim_{h \to 0} \frac{-1}{1+h}$$

$$= -1$$

And we know $f(1) = 1$, so the tangent line passes through $(1, 1)$, together with slope being 1, we see the equation of the tangent line of $f(x) = \frac{1}{x}$ at $x = 1$ is
$$y - 1 = -(x - 1)$$

Apart from the slope of a function, there are other situation involving taking a limit of the quotient
$$\frac{f(a+h) - f(a)}{h}$$
A good example is the instant velocity.

Consider an object moving on the real number line. Define the displacement function $s = f(t)$, which indicates the position of the particle at each time t.

We know that the average velocity of the object between time $t=a$ and time $t=a+h$ is

$$\frac{\Delta s}{\Delta t} = \frac{f(a+h)-f(a)}{h}$$

If we take h to be smaller and small, which means the time interval is shorter and shorter, we will eventually obtain a limit, if exists, as $h \to 0$. We define

$V(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ to be the instant velocity of this object at time $t=a$.

The intuition of instant velocity $V(a)$ is that, if we consider the movement of the object around a small time interval at $t=a$, then the average velocity on this small time interval is close to $V(a)$. As the time interval is smaller and smaller, the average velocity is closer and closer to $V(a)$.

Example. A ball is falling from 1000 m above the ground. The distance fallen after t seconds is $4.9t^2$.

(a). What is the velocity of the ball after 5 seconds?

(b). How fast is the ball travelling when it hit the ground?

$$V(a) = \lim_{h\to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h}$$

$$= \lim_{h\to 0} \frac{9.8ah + 4.9h^2}{h}$$

$$= \lim_{h\to 0} (9.8a + 4.9h)$$

$$= 9.8a$$

(a) $V(5) = 9.8\times 5 = 49$ m/s

(b) We need to figure out how long does it take for the ball to reach the ground:

$$4.9t^2 = 1000 \Rightarrow t = \sqrt{\frac{1000}{4.9}} = \sqrt{\frac{10000}{49}} = \frac{100}{7}$$

So the velocity when it hit the ground is

$$v\left(\frac{100}{7}\right) = 9.8 \times \frac{100}{7} = 140 \text{ m/s}$$

If we compare the discussion of the slope of the graph of a function and the instant velocity, we find they can be related in the following way:

We draw the graph of the displacement function $s=f(t)$. then the slope at $t=a$ equals to the instant velocity $v(a)$. The reason is both of them satisfy a same formula involving taking limits, which represents the rate of change of the same quantity.

So we have seen that the concept of "rate of change" plays an important role when studying the behavior of a function near a point. We now make the following general definition:

Definition. The derivative of a function f at a number a, denoted by $f'(a)$, is:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

if this limit exists.

Example. Find the derivative of $f(x) = x^2-8x+9$ at a number b.

$$f'(b) = \lim_{h\to 0} \frac{f(b+h)-f(b)}{h} = \lim_{h\to 0} \frac{[(b+h)^2-8(b+h)+9]-[b^2-8b+9]}{h}$$

$$= \lim_{h\to 0} \frac{b^2+2bh+h^2-8b-8h+9-b^2+8b-9}{h}$$

$$= \lim_{h\to 0} \frac{2bh + h^2 - 8h}{h}$$

$$= \lim_{h\to 0} (2b - 8 + h)$$

$$= 2b - 8$$

Example. Let $f(x) = |x|$.

$$f'(0) = \lim_{h\to 0} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Note that $\frac{|h|}{h} = \begin{cases} 1 & \text{if } h>0 \\ -1 & \text{if } h<0 \end{cases}$

so $\lim_{h\to 0^+} \frac{|h|}{h} = 1$, $\lim_{h\to 0^-} \frac{|h|}{h} = -1$, $1 \neq -1$. which indicates $\lim_{h\to 0} \frac{|h|}{h}$ does not exists, so $f'(0)$ doesn't exist.

Using the language of derivative, we can define the tangent line in the following way:

Definition. The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$.

Example. Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at $(3, -6)$

Let $f(x) = x^2 - 8x + 9$. By our previous example, we know $f'(b) = 2b - 8$. So $f'(3) = 2 \times 3 - 8 = -2$.

So the equation of the tangent line is $y + 6 = -2(x-3)$

As we mentioned as the motivation of the concept of derivative, derivative $f'(a)$ describes the instantaneous rate of change of $f(x)$ with respect to x at $x=a$. So we can summarize as follows:

The derivative $f'(a)$ is the slope of the tangent line to the graph of $y=f(x)$ at $(a, f(a))$, and it represents the instantaneous rate of change of $y=f(x)$ with respect to x when $x=a$.

As an application, if we consider the displacement (or position) function $s=f(t)$, then $f'(a)$ is the velocity at time $t=a$. Note the velocity can be a positive value or a negative value. The sign of the velocity tells us about to which direction the object is travelling. The absolute value $|v(a)|$ is called the speed, which measures how fast the object is travelling at time $t=a$.