INFINITY

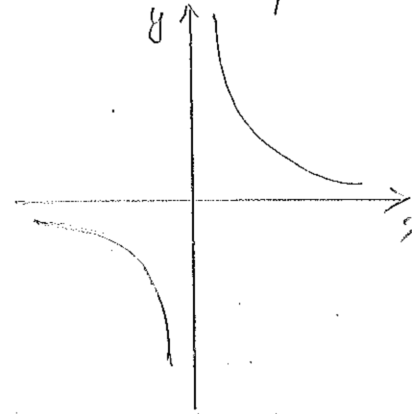
Definition. The notation $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (on either side of a) but not equal to a.

Sometimes people also write: $f(x) \to \infty$ as $x \to a$.

Definition. The notation $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a (on either side of a) but not equal to a

Example. $f(x) = \frac{1}{x}$.

$\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$



Example. $f(x) = \frac{1}{x^2}$.

Observe that as $x \to 0$, $x^2 \to 0^+$

so $\lim_{x \to 0} \frac{1}{x^2} = \lim_{y \to 0^+} \frac{1}{y} \quad (y = x^2)$

$= +\infty$

Definition. The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

- $\lim_{x \to a} f(x) = \infty$ · $\lim_{x \to a^+} f(x) = \infty$ · $\lim_{x \to a^-} f(x) = \infty$
- $\lim_{x \to a} f(x) = -\infty$ · $\lim_{x \to a^+} f(x) = -\infty$ · $\lim_{x \to a^-} f(x) = -\infty$

Example. Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$. Also find the vertical assymptote for $y = \frac{2x}{x-3}$

If x approaches 3 from right side, $x-3$ is a small positive number, so $\frac{1}{x-3}$ is a large positive number, and $2x$ approaches 6, we get the quotient $\frac{2x}{x-3}$ goes to ∞. $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$

If x approaches 3 from left side, $x-3$ is a small negative number, so $\frac{1}{x-3}$ is a large negative number, and $2x$ approaches 6, we get the quotient $\frac{2x}{x-3}$ goes to $-\infty$, $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$

The vertical assymptote is $x=3$.

Example. $f(x) = \tan x = \frac{\sin x}{\cos x}$

As $x \to (\frac{\pi}{2})^-$, $\cos x \to 0^+$, and $\sin x \to 1$.
so $\tan x \to \infty$. $\lim_{x\to(\frac{\pi}{2})^-} \tan x = \infty$

As $x \to (\frac{\pi}{2})^+$, $\cos x \to 0^-$. and $\sin x \to 1$.
so $\tan x \to -\infty$. $\lim_{x\to(\frac{\pi}{2})^+} \tan x = -\infty$

We see $x = \frac{\pi}{2}$ is a vertical assymptote for $f(x) = \tan x$

$f(x) = \tan x$ is a periodic function with period π.
So the behavior of the function will repeat with respect to the period π, we get

$\lim_{x\to(\frac{\pi}{2}+k\pi)^+} \tan x = \infty$, $\lim_{x\to(\frac{\pi}{2}+k\pi)^-} \tan x = -\infty$, and

$x = \frac{\pi}{2} + k\pi$ is a vertical assymptote.
where k is an integer.

Definition. Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the value of $f(x)$ can be made arbitrarily close to L as we take x sufficiently large.

Example. $f(x) = \frac{1}{x}$ $\quad \lim_{x\to\infty} f(x) = 0$, since when x is sufficiently large, $\frac{1}{x}$ is very close to 0.

Example. $f(x) = \frac{x^2-1}{x^2+1}$

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{(x^2+1)-2}{x^2+1} = \lim_{x\to\infty}\left(1-\frac{2}{x^2+1}\right) = 1 - \lim_{x\to\infty} \frac{2}{x^2+1}$$
$$= 1 - 0$$
$$= 1$$

Similarly, we can define $\lim_{x\to-\infty} f(x) = L$ in a same fashion, just replace "x is sufficiently large" by "x is sufficiently negatively large."

Example. $f(x) = \frac{1}{x}$, $\lim_{x\to-\infty} f(x) = 0$, since when x is sufficiently negatively large, $\frac{1}{x}$ is very close to 0.

Proposition. n is a positive integer, then:

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Example. Evaluate $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$

$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}(3x^2-x-2)}{\frac{1}{x^2}(5x^2+4x+1)}$$
$$= \lim_{x\to\infty} \frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$
$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{2}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{4}{x} + \lim_{x\to\infty}\frac{1}{x^2}} = \frac{3}{5}$$

Example. Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x) = \lim_{x\to\infty}(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{(\sqrt{x^2+1})^2-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

$$= 0$$

Example. Evaluate $\lim_{x\to\infty}\sin\frac{1}{x}$

Since the sine function is continuous.

$\lim_{x\to\infty}\sin\frac{1}{x} = \sin(\lim_{x\to\infty}\frac{1}{x}) = \sin 0 = 0$

Definition. We write $\lim_{x\to\infty}f(x)=\infty$ to indicate the values of $f(x)$ become large as x becomes large.

Similarly we also have $\lim_{x\to-\infty}f(x)=\infty$, $\lim_{x\to\infty}f(x)=-\infty$, $\lim_{x\to-\infty}f(x)=-\infty$

Example. $\lim_{x\to\infty}x^2-x = \lim_{x\to\infty}x(x-1) = \left(\lim_{x\to\infty}x\right)\left(\lim_{x\to\infty}(x-1)\right) = \infty$.

NOTE. We have no algorithm for $\infty-\infty$ or $\frac{\infty}{\infty}$.

Example. $\lim_{x\to\infty}\frac{x^2+x}{3-x} = \lim_{x\to\infty}\frac{\frac{1}{x}(x^2+x)}{\frac{1}{x}(3-x)} = \lim_{x\to\infty}\frac{x+1}{\frac{3}{x}-1}$

Since $\lim_{x\to\infty}(x+1)=\infty$ and $\lim_{x\to\infty}(\frac{3}{x}-1)=-1$,

we get $\lim_{x\to\infty}\frac{x+1}{\frac{3}{x}-1} = -\infty$