CONTINUITY

Definition. A function f is continuous at a number b if
$$\lim_{x\to b} f(x) = f(b).$$

The above definition means the following conditions need to be satisfied if a function f is continuous at b:

① b is in the domain of f. i.e. $f(b)$ is defined

② The limit $\lim_{x\to b} f(x)$ exists

③ $\lim_{x\to b} f(x) = f(b)$

Examples. (i). $f(x) = \frac{\sin x}{x}$ is NOT continuous at $x=0$ since it's not defined there.

(ii) $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is NOT continuous at $x=0$.

Since $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = 0$, which indicates $\lim_{x\to 0} f(x)$ doesn't exist.

(iii) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x=2 \end{cases}$ is NOT continuous at $x=2$,

Since $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2}$
$$= \lim_{x\to 2} (x+1) = 3$$

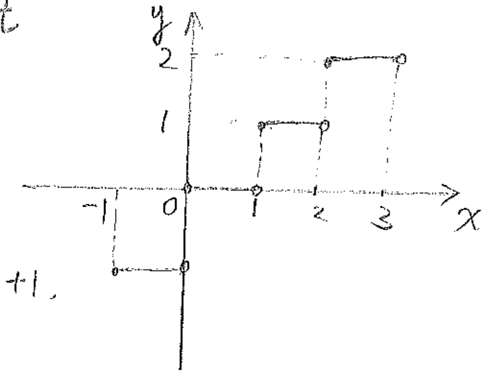
but $f(2) = 1 \neq 3 = \lim_{x\to 2} f(x)$.

(iv). $f(x) = |x|$ is continuous at $x = 0$:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0, \quad \text{and } f(0) = |0| = 0.$$

So $\lim\limits_{x \to 0} f(x) = f(0)$.

(v). Consider the function $f(x) = [[x]]$, where $[[x]]$ is defined to be the maximum integer that is not bigger than x. This function is called the Greatest Integer Function.



One way to describe this function is

$$f(x) = [[x]] = n \quad \text{for } n \le x < n+1,$$
where n is an integer.

At each integer point n, we see

$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [[x]] = n-1 \quad \text{while}$$

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [[x]] = n.$$

so $\lim\limits_{x \to n} f(x)$ doesn't exist, $f(x)$ is not continuous at the integer points.

In the last example (v), we notice that although $f(x)$ is not continuous at integer points, we still have

$$\lim_{x \to n^+} f(x) = f(n).$$

So if we only concentrate on the behavior of $f(x)$ at one of the two sides of a given point, we can define the one-side continuity:

Definition. A function $f(x)$ is continuous from the right at a number b if $\lim_{x\to b^+} f(x) = f(b)$.

A function $f(x)$ is continuous from the left at a number b if $\lim_{x\to b^-} f(x) = f(b)$

Example. So for $f(x) = [[x]]$, $f(x)$ is continuous from the right at integer points, since $\lim_{x\to n^+} f(x) = n = f(n)$

But $f(x) = [[x]]$ is not continuous from the left at integer points, since $\lim_{x\to n^-} f(x) = n-1 \neq f(n)$

By the definitions, we see $f(x)$ is continuous at b if and only if $f(x)$ is continuous from both left and right at b.

Definition. A function $f(x)$ is continuous on an interval if it is continuous at each point of the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left)

Example. Show that $f(x) = 1-\sqrt{1-x^2}$ is continuous on $[-1, 1]$.

First, for any $-1<a<1$,

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(1-\sqrt{1-x^2}\right) = 1-\lim_{x\to a}\sqrt{1-x^2} = 1-\sqrt{\lim_{x\to a}(1-x^2)} = 1-\sqrt{1-a^2} = f(a).$$

Next, $\lim_{x\to -1^+} f(x) = 1 = f(-1)$, $\lim_{x\to 1^-} f(x) = 1 = f(1)$.

So $f(x)$ is continuous on $[-1, 1]$.

Theorem. If f & g are continuous at b and c is a constant, then the following functions are also continuous at b:

(i). $f+g$ (ii) $f-g$ (iii) cf (iv) fg (v) $\frac{f}{g}$ if $g(b)\neq 0$

Theorem. (a). Any polynomial is continuous everywhere.
(b). Any rational function is continuous wherever it is defined: that is, it's continuous on its domain.
(c) Any root function is continuous wherever it is defined
(d). Any trigonometric function is continuous wherever it is defined.

Example. On what intervals is each function continuous?
(a). $f(x) = x^3 - 3x$ (b) $g(x) = g(x) = \frac{3x+1}{x^2+x-2}$
(c). $h(x) = \sqrt{x} + \sin x + \frac{1}{x-1}$

(a). $f(x)$ is a polynomial, so it's continuous everywhere. i.e. on $(-\infty, +\infty)$.

(b). $g(x)$ is a rational function, it's domain is $x^2+x-2\neq 0$ i.e. $(x+2)(x-1)\neq 0$, $x\neq -2$ and $x\neq 1$. So, it's continuous on $(-\infty, -2)$, $(-2, 1)$ and $(1, +\infty)$.

(c). $h(x)$ is a sum of a root function, a trigonometric function and a rational function, so it's continuous on any point in its domain. The domain is $[0,1)\cup(1,+\infty)$, so $h(x)$ is continuous on the intervals $[0,1)$ and $(1,+\infty)$

Theorem. (i) If f is continuous at b and $\lim_{x\to a} g(x) = b$, then:
$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$$

(ii). If g is continuous at a and f is continuous at $g(a)$, then the composite function $f\circ g(x) = f(g(x))$ is continuous at a.

Example. Where are the following functions continuous?

(a). $h(x) = \sin x^2$. (b) $f(x) = \frac{1}{\sqrt{x^2+1}-4}$

(a) $h(x) = \sin x^2$ is the composition of $f(x) = \sin x$ and $g(x) = x^2$: $h(x) = f(g(x))$.

g is continuous on $(-\infty, +\infty)$, and f is also continuous everywhere, so $h(x)$ is continuous on $(-\infty, +\infty)$

(b). $f(x) = g_1 \circ g_2 \circ g_3 \circ g_4 (x)$, where $g_1(x) = \frac{1}{x}$, $g_2(x) = x-4$, $g_3(x) = \sqrt{x}$, $g_4(x) = x^2+1$.

each of them is continuous on its domain, so the composition $f(x)$ is also continuous on its domain.

Theorem. (The Intermediate Value Theorem)
Suppose that f is continuous on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

Example. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

$f(1) = -1 < 0$, $f(2) = 12 > 0$.
so $f(1) < 0 < f(2)$, by the Intermediate Value Theorem, there's $c \in (1, 2)$ such that $f(c) = 0$.