We have discussed in the previous section that the limit $\lim_{x\to b} f(x)$ doesn't depend on the value at $x=b$, but only depends on the values near $x=b$ excluding $x=b$. So we have the following result:

Proposition. If $f(x)=g(x)$ for all x near b but $x\neq b$, then $\lim_{x\to b} f(x) = \lim_{x\to b} g(x)$ if the limit exists.

Example. $f(x) = \begin{cases} x^3, & x\neq 0 \\ 8, & x=0 \end{cases}$ find $\lim_{x\to 0} f(x)$.

Let $g(x)=x^3$. Then $f(x)=g(x)$ for all $x\neq 0$.

we therefore can compute by

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} x^3 = 0^3 = 0$$

Recall that we have learned the concepts of left limit and right limit. There is a way to find the limit by considering left limit and right limit:

Proposition. $\lim_{x\to b} f(x) = L$ if and only if $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} f(x) = L$

Note that all the previous limit laws also apply to one-side limits.

Example. Compute $\lim_{x\to 0} |x|$

We know $|x| = \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$.

so $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$

We see $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x| = 0$, so $\lim_{x\to 0} |x| = 0$

Example. $f(x)=\begin{cases} x^2+1, & x\geq 0 \\ -3x, & x<0 \end{cases}$ show that $\lim\limits_{x\to 0} f(x)$ doesn't exist.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2+1) = 0^2+1=1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-3x) = -3\cdot 0 = 0.$$

$\lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$, so $\lim\limits_{x\to 0} f(x)$ doesn't exist.

Theorem. If $f(x) \leq g(x)$ for all x near b (except possibly at b) and $\lim\limits_{x\to b} f(x)$, $\lim\limits_{x\to b} g(x)$ both exist, then $\lim\limits_{x\to b} f(x) \leq \lim\limits_{x\to b} g(x)$.

The above theorem implies the following theorem:

Theorem. (The Squeeze Theorem / The Sandwich Theorem)
If $f(x) \leq g(x) \leq h(x)$ when x is near b (except possibly at b), and $\lim\limits_{x\to b} f(x) = \lim\limits_{x\to b} h(x) = L$, then $\lim\limits_{x\to b} g(x) = L$

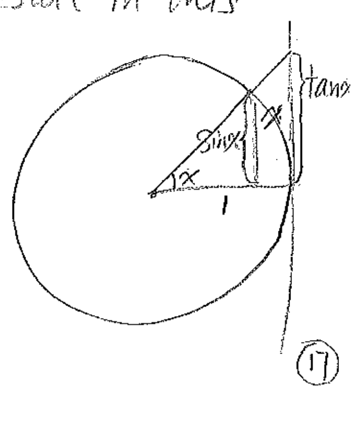
Example. Compute $\lim\limits_{x\to 0} x^2\cdot \sin\frac{1}{x}$.

Since $-1 \leq \sin\frac{1}{x} \leq 1$, $-x^2 \leq x^2\sin\frac{1}{x} \leq x^2$.
we know $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$.

By The Squeeze Theorem, $\lim\limits_{x\to 0} x^2\sin\frac{1}{x} = 0$.

Example. In this example, we will show an important result in this course:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$



This follows from the geometric observation:

when x is near 0 and $x>0$: $\sin x < x < \tan x = \frac{\sin x}{\cos x}$

This implies $\cos x < \frac{\sin x}{x} < 1$

We know $\lim_{x\to 0^+} \cos x = 1$, so by The Squeeze Theorem,

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

Similarly we can show $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$, so we conclude

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Example. Compute $\lim_{x\to 0} \frac{\sin 7x}{3x}$

$$\lim_{x\to 0} \frac{\sin 7x}{3x} = \lim_{x\to 0} \frac{\sin 7x}{7x} \cdot \frac{7x}{3x} = \frac{7}{3} \lim_{x\to 0} \frac{\sin 7x}{7x}$$

$$= \frac{7}{3} \lim_{y\to 0} \frac{\sin y}{y} \quad (\text{let } y = 7x)$$

$$= \frac{7}{3} \cdot 1$$

$$= \frac{7}{3}$$

Example. Compute $\lim_{x\to 0} \frac{\cos x - 1}{|x|}$

This is a piecewise defined function: $\frac{\cos x - 1}{|x|} = \begin{cases} \frac{\cos x - 1}{x}, & x > 0 \\ \frac{1 - \cos x}{x}, & x < 0 \end{cases}$

$$\lim_{x\to 0^+} \frac{\cos x - 1}{|x|} = \lim_{x\to 0^+} \frac{\cos x - 1}{x}$$

$$= \lim_{x\to 0^+} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$$

$$= \lim_{x\to 0^+} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}$$

$$= -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \frac{\sin x}{\cos x + 1} = -1 \cdot 0 = 0$$

$$\lim_{x\to 0^-} \frac{\cos x - 1}{|x|} = \lim_{x\to 0^-} \frac{1-\cos x}{x} = \lim_{x\to 0^-} \frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0^-} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0^-} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0^-} \frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}$$

$$= \left(\lim_{x\to 0^-}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^-}\frac{\sin x}{1+\cos x}\right)$$

$$= 1\cdot 0$$

$$= 0$$

Since $\lim_{x\to 0^+} \frac{\cos x - 1}{|x|} = \lim_{x\to 0^-}\frac{\cos x - 1}{|x|} = 0$,

we conclude $\lim_{x\to 0}\frac{\cos x - 1}{|x|} = 0$

Example | Compute $\lim_{x\to 0} \frac{\sin x + \tan x}{\sin x + x}$

$$\lim_{x\to 0}\frac{\sin x - \tan x}{\sin x + x} = \lim_{x\to 0}\frac{\sin x + \frac{\sin x}{\cos x}}{\sin x + x}$$

$$= \lim_{x\to 0}\frac{\frac{\sin x}{x} + \frac{\sin x}{x}\cdot\frac{1}{\cos x}}{\frac{\sin x}{x} + 1}$$

$$= \frac{\lim_{x\to 0}\frac{\sin x}{x} + \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}}{\lim_{x\to 0}\frac{\sin x}{x} + 1}$$

$$= \frac{1 + 1\cdot 1}{1+1}$$

$$= 1$$