LIMIT OF A FUNCTION.

Definition. Suppose a function $f(x)$ is defined on some open interval containing b, except possibly at b itself.
If $f(x)$ approaches arbitrarily close to a real number L as x is taken sufficiently close to b but not equal to b, we define the limit of $f(x)$ as x approaches b equals L, and write it as

$$\lim_{x \to b} f(x) = L$$

Also, sometimes people write "$f(x) \to L$ as $x \to b$".

Remark: By the definition, we see that $\lim_{x\to b} f(x)$ only depends on the behavior of $f(x)$ near $x=b$, but not at $x=b$. In other words, the value of $f(b)$ won't affect $\lim_{x\to b} f(x)$.

Example. Let's study $f(x) = \frac{x-1}{x^2-1}$. Observe that the function is not defined at $x=1$, but we will see $\lim_{x\to 1} f(x)$ exists.

At first glance, if x approaches 1, then both $x-1$ and x^2-1 will tend to 0. But we have no idea about the value $\frac{0}{0}$, which is not defined.

If we try some values near 1 (as shown in the textbook), we will see that as $x \to 1$, $f(x) \to \frac{1}{2}$.

We can find the reason as follows:
$f(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$ (since $x \neq 1$)
so if we let $x \to 1$, $f(x) = \frac{1}{x+1} \longrightarrow \frac{1}{1+1} = \frac{1}{2}$.
We conclude $\lim_{x\to 1} f(x) = \frac{1}{2}$.

Limit does NOT always exists. Sometimes as $x \to b$, we'll find that $f(x)$ doesn't converge to any fixed number.

In this case, we say the limit $\lim_{x \to b} f(x)$ doesn't exist

Example. $H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

We see as t approaches from the right to 0, $f(t)$ is constantly 1, and as t approaches from the left to 0, $f(t)$ is constantly 0. So there doesn't exist a number L such that $f(t)$ is approaching L as t approaches 0. The limit $\lim_{t \to 0} H(t)$ doesn't exist.

In the above example, we see although the limit $\lim_{t \to 0} H(t)$ doesn't exist, when we only concentrate on one side of the neighbourhood near 0, the function will approach to a fixed value. This motivates the next definition.

Definition. We say the left-hand limit (or left limit) of $f(x)$ as x approaches b is equal to L if $f(x)$ will be arbitrarily close to L when we take x sufficiently close to b and $x < b$. We write $\lim_{x \to b^-} f(x) = L$.

We say the right-hand limit (or right limit) of $f(x)$ as x approaches b is equal to L if $f(x)$ will be arbitrarily close to L when we take x sufficienty close to b and $x > b$. We write $\lim_{x \to b^+} f(x) = L$.

Example So for the function $H(t)$ in the previous example, $\lim_{x \to 0^+} H(t) = 1$ and $\lim_{x \to 0^-} H(t) = 0$.

We can also take $b = +\infty$ or $b = -\infty$ to talk about the behavior of $f(x)$ when $x \to +\infty$ or $x \to -\infty$.

Definition. $f(x)$ approaches L as x approaches $+\infty$ if $f(x)$ will be arbitrarily close to L when x is sufficiently big. We write $\lim\limits_{x \to +\infty} f(x) = L$.

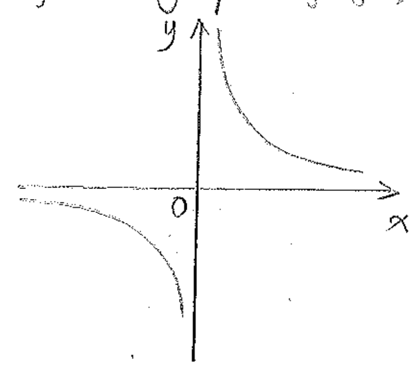
$f(x)$ approaches L as x approaches $-\infty$ if $f(x)$ will be arbitrarily close to L when $-x$ is sufficiently big. We write $\lim\limits_{x \to -\infty} f(x) = L$

Example. $f(x) = \frac{1}{x}$.

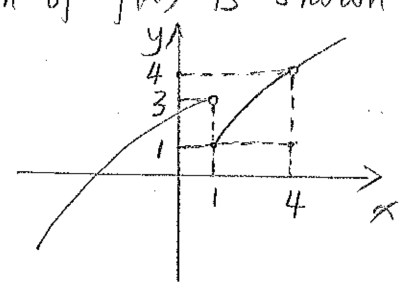
As x is sufficiently big, $\frac{1}{x}$ will be sufficiently small, approaching 0 as x approaching $+\infty$, so $\lim\limits_{x \to +\infty} f(x) = 0$

Similarly, we can also see that $\lim\limits_{x \to -\infty} f(x) = 0$

Both limits agree with the shape of the graph of $f(x)$



Example. If the graph of $f(x)$ is shown as below.



We see $\lim\limits_{x \to 1^-} f(x) = 3$, $\lim\limits_{x \to 1^+} f(x) = 1$, $\lim\limits_{x \to 1} f(x)$ does not exist.

$\lim\limits_{x \to 4} f(x) = 4$

CALCULATING LIMITS.

Proposition (Limit Laws). Suppose c is a constant, and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

① $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

② $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

③ $\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$

④ If $\lim_{x\to a} g(x) \neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

⑤ If n is a positive integer, $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$.

⑥ $\lim_{x\to a} c = c$

⑦ $\lim_{x\to a} x = a$.

⑧ If n is a positive integer,
$\lim_{x\to a} x^n = a^n$ and $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

⑨ If n is a positive integer, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

The above Proposition provides a convenient way to compute the limit of a function.

Example. Compute $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)} = \frac{(\lim_{x\to -2} x^3) + (\lim_{x\to -2} 2x^2) - \lim_{x\to -2} 1}{(\lim_{x\to -2} 5) - (\lim_{x\to -2} 3x)}$$

$$= \frac{(-2)^3 + 2\cdot(-2)^2 - 1}{5 - 3\times(-2)} = -\frac{1}{11}$$

Example. $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^2-3x+2} = \lim\limits_{x\to 1} \dfrac{(x-1)^2}{(x-1)(x-2)} = \lim\limits_{x\to 1} \dfrac{x-1}{x-2} = \dfrac{\lim\limits_{x\to 1}(x-1)}{\lim\limits_{x\to 1}(x-2)}$

$= 0$

Note that the limit

$\lim\limits_{x\to 1}(x^2-3x+2)=0$, so we CANNOT write $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^2-3x+2} = \dfrac{\lim\limits_{x\to 1}(x^2-2x+1)}{\lim\limits_{x\to 1}(x^2-3x+2)}$

In many cases, if the given function is "good", there is a much faster way to find the limit.

Proposition. If f is a polynomial or a rational function, and b is in the domain of f, then:

$$\lim_{x\to b} f(x) = f(b)$$

The above is also true for $\sin x$ and $\cos x$.

Example. $\lim\limits_{x\to\pi} x^2 \sin x = \left(\lim\limits_{x\to\pi} x^2\right)\left(\lim\limits_{x\to\pi} \sin x\right) = \pi^2 \cdot \sin\pi = 0$

Example. $\lim\limits_{x\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2} = \lim\limits_{x\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2} \cdot \dfrac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

$$= \lim_{x\to 0} \frac{t^2}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\sqrt{0^2+9}+3}$$

$$= \frac{1}{6}$$