POPULAR FUNCTIONS

Definition. A function P is a polynomial if P is of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

with the leading coefficient $a_n \neq 0$.
we call n to be the degree of $P(x)$.

When $n = 0$, $P(x) = a_0$ is a constant function, which assigns a_0 to every number in the domain

When $n=1$, $P(x)$ is a linear function, and we usually write it as $P(x) = mx + b$. (the slope-intercept form)
Its graph is a straight line, with slope m and y-intercept b.

When $n=2$ $P(x)$ is a quadratic function, and we usually write it as $P(x) = ax^2 + bx + c$. $(a \neq 0)$
Its graph is a parabola, which opens upward if $a > 0$ and opens downward if $a < 0$.

When P is of the form $P(x) = x^n$. (n is a positive integer), we obtain a special case of the power function.

Proposition. $f(x) = x^n$ is an even function if n is even.
$f(x) = x^n$ is an odd function if n is odd.

$f(x) = x^{\frac{1}{n}}$ is called the root function. which is defined as $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$.

$f(x) = x^{-1}$ is the reciprocal function, which is defined as $f(x) = x^{-1} = \frac{1}{x}$

Definition. A rational function f is the ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

Its domain is the set of x satisfying $Q(x) \neq 0$

In formal mathematical expression: $\{x \in \mathbb{R} \mid Q(x) \neq 0\}$

Example. Are $f(x) = \frac{x+1}{x-2}$ and $g(x) = \frac{x^2+3x+2}{x^2-4}$ having the same domain?

Although you may observe that $g(x) = \frac{(x+1)(x+2)}{(x-2)(x+2)}$,

$f(x)$ and $g(x)$ have different domains.

The domain of f is $\{x \in \mathbb{R} \mid x \neq 2\}$, while the domain of g is $\{x \in \mathbb{R} \mid x \neq 2 \text{ and } x \neq -2\}$

You cannot cancel the term $(x+2)$ for $g(x)$, otherwise you will change the domain of $g(x)$.

Trigonometric Functions.

For definitions of trigonometric functions and basic properties, you may review high-school math or precalculus.

Proposition. $f(x) = \sin x$ and $g(x) = \cos x$ both have domain $\mathbb{R}$ and range $[-1, 1]$. They are also periodic functions:

$\sin(x+2\pi) = \sin x$, $\cos(x+2\pi) = \cos x$

$h(x) = \tan x$ is not defined when $x = \frac{\pi}{2} + k\pi$, where k is an integer. $\tan x$ has range $\mathbb{R}$.

Proposition. $\sin x$ and $\tan x$ are odd functions. $\cos x$ is an even function.

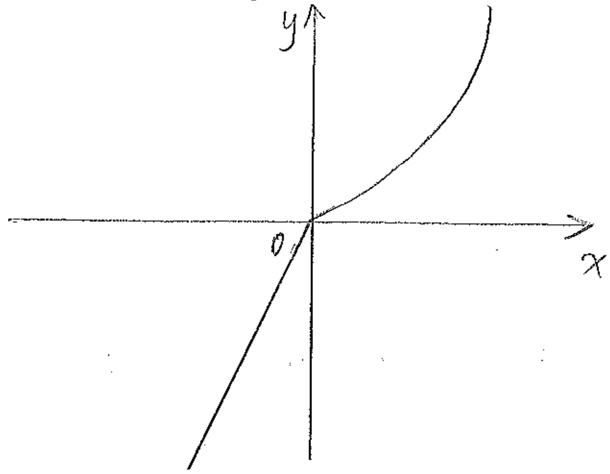
Definition. Piecewise Defined Functions:

A function is piewise defined if it has different formulas in different parts of the domain.

Example. $f(x)=\begin{cases} x^2 & \text{if } x \geq 0 \\ 3x & \text{if } x<0 \end{cases}$

$f(3)=3^2=9$

$f(-3)=3\cdot(-3)=-9$

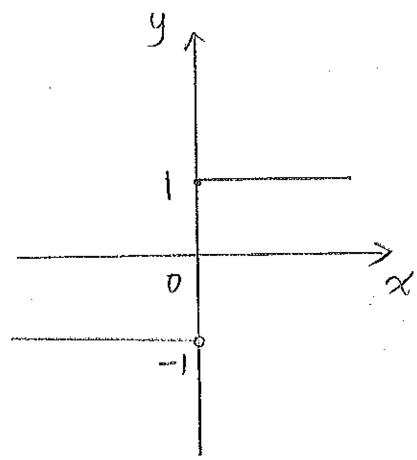


Example. The absolute value function $f(x)=|x|$ can be regarded as a piecewise defined function:

$$f(x)=|x|=\begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$$

Example. Step Function.

$$f(x)=\begin{cases} 1 & x \geq 0 \\ -1 & x<0 \end{cases}$$



We also need to review the logarithmic function. Given a real number a such that $a>0$ and $a \neq 1$.

We define $y=\log_a x \Longleftrightarrow x=a^y$

The logarithmic function is $f(x)=\log_a x$, which has the following properties:

① . $f(1) = \log_a 1 = 0$

② $f(x) = \log_a x$ is an increasing function if $a > 1$.

$f(x) = \log_a x$ is a decreasing function if $0 < a < 1$.

③ . $f(x_1 x_2) = f(x_1) + f(x_2)$

i.e. $\log_a(x_1 x_2) = \log_a x_1 + \log_a x_2$

④ Change of base formula:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

In the case when $a = e$, the Euler's number, we write $f(x) = \ln x$ instead of $f(x) = \log_e x$.

So ④ in particular can be written as

$$\log_a b = \frac{\ln b}{\ln a} \quad (\text{when we take } c = e)$$

⑤ The domain of $f(x) = \log_a x$ is the set of all positive numbers $(0, +\infty)$, and the range of $f(x) = \log_a x$ is the set of all real numbers $\mathbb{R}$.

Example. Find the domain of the function $f(x) = \ln \sin x$

We find this is the composition of $g(x) = \sin x$ and $h(x) = \ln x$

i.e. $f = h \circ g$.

the domain of $\ln x$ is $(0, +\infty)$, so we need

$\sin x > 0$.

which corresponds to $0 + 2k\pi < x < \pi + 2k\pi$, $k \in \mathbb{Z}$