FUNCTIONS

A Brief Review

Definition. Given two sets D and E, a function f is a rule that assigns to each element x in D exactly one element $f(x)$ in E.

- D is the domain of the function.
- The range of the function is the set of all $f(x)$ as x varies in D.

In most of this course, we will consider D and E to be subsets of the real numbers $\mathbb{R}$.

The most common way to represent a function is to write out the algebraic expression of $f(x)$ in terms of x

Example. ① The Identity Function: $f(x)=x$ defined on $\mathbb{R}$.
② The Absolute Value Function: $f(x)=|x|$ defined on $\mathbb{R}$
③ $f(x)=\frac{1}{x-1}$ defined on $(-\infty,-1)\cup(1,+\infty)$

When an algebraic expression is given, we can find the value $f(x_0)$ assigned to x_0 by evaluating the expression at x_0

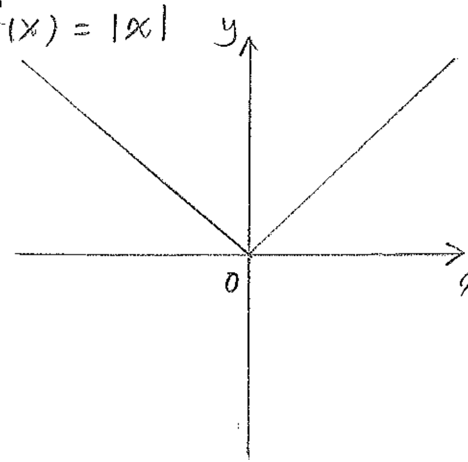
Example. If $f(x)=x^2+1$, then $f(3)=3^2+1=10$

Sometimes the function is defined descriptively by words. In such situation, we usually try to translate it into the algebraic expression before applying mathematical tools to solve problems.

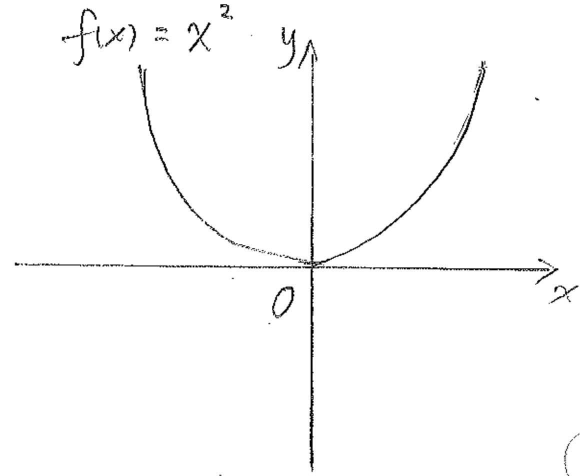
Example. The volume of a cube is the third power of its edge length. So we can let the length of the edge to be x, then the corresponding volume is a function of x given by $f(x) = x^3$

We can also describe functions by graphs on the x-y plane. Given a function f, its graph is the set of points $(x, f(x))$ on the xy-plane. (We use the Cartisian Coordinate)

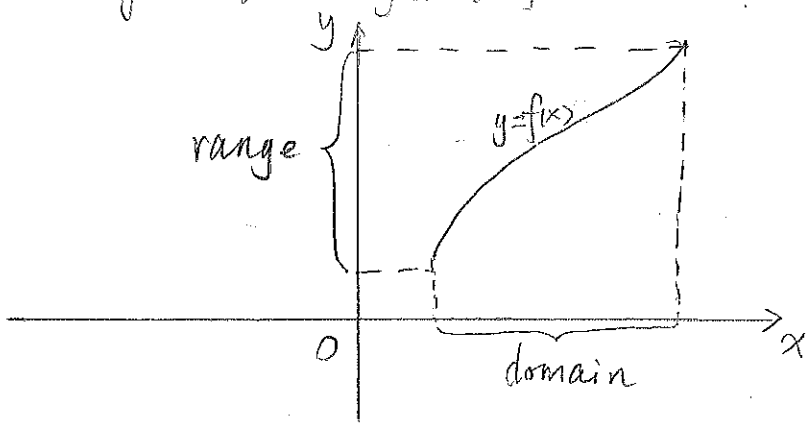
Example. $f(x) = |x|$



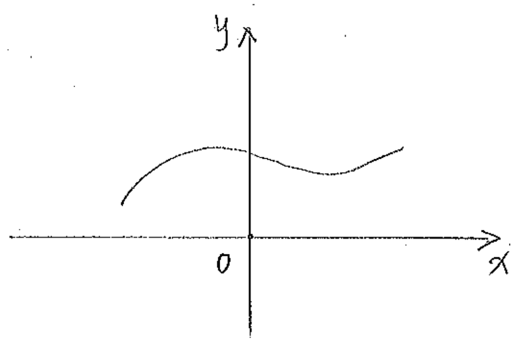
$f(x) = x^2$



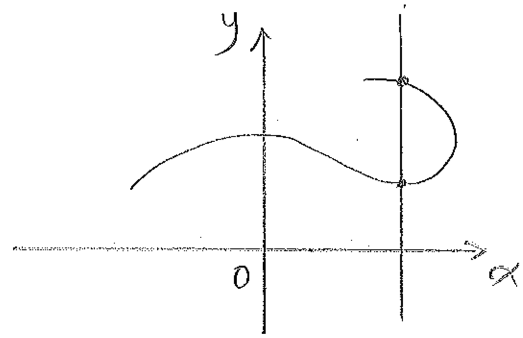
Proposition. Given the graph of a function f, its projection on the x-axis gives the domain of f, and its projection on the y-axis gives the range of f



Proposition (The vertical Line Test). A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.



Graph of a function



Not Graph of a function

Definition. The domain of a function $f(x)$ is the set of numbers from which x can take value of.
If the domain is not specified for a given function, we regard the domain to be the set of all values for which the function gives a unique value.

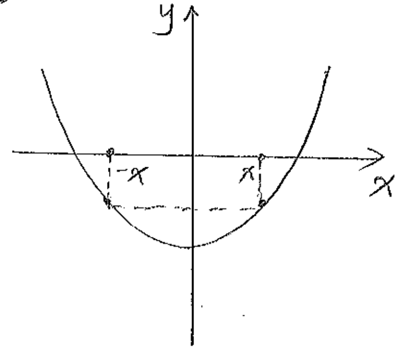
Example. Find the domain of $f(x) = \frac{1}{x-1}$:

$x-1$ is at the denominator, so we need $x-1 \neq 0$, i.e. $x \neq 1$.
So the domain is $(-\infty, 1) \cup (1, +\infty)$

Example. Find the domain of $f(x) = \sqrt{x-1}$
$x-1$ is under square root, so we need $x-1 \geq 0$, i.e. $x \geq 1$.
So the domain is $[1, +\infty)$

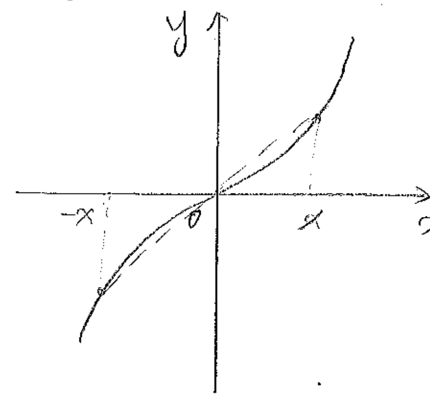
Definition. f is an even function if for any x in the domain, $f(x) = f(-x)$.

The graph of an even function is symmetric respect to the y-axis.



Definition. f is an odd function if for any x in the domain
$f(-x) = -f(x)$.

The graph of an odd function is symmetric with respect to the origin.



Examples (1). $f(x) = x^3$ is an odd function:

$$f(-x) = (-x)^3 = (-1)^3 x^3 = -x^3 = -f(x)$$

(2) $f(x) = x^2$ is an even function:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

(3) For any real number C, the constant function $f(x) \equiv C$ is an even function;

$$f(-x) \equiv C \equiv f(x)$$

(4). $f(x) = 2x - x^2$.

$$f(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

we see $f(-x) \neq f(x)$, $f(-x) \neq -f(x)$
so f is neither odd nor even.

Definition.
- A function f is increasing on an interval I if for any $x_1 < x_2$ in I, $f(x_1) < f(x_2)$.
- A function f is decreasing on an interval I if for any $x_1 < x_2$ in I, $f(x_1) > f(x_2)$.

Later in this course we will learn to use calculus to tell if a function is increasing or decreasing.

Definition: Given two functions f and g, the composition function $f\circ g$ is defined by $(f\circ g)(x) = f(g(x))$.

Example. $f(x) = \sqrt{x+1}$, $g(x) = x^2$, then

$$f\circ g(x) = f(g(x)) = \sqrt{g(x)+1} = \sqrt{x^2+1}$$

We need to be careful when calculating the domain of $f\circ g$, which may be different from that of g.

Example. $f(x) = \sqrt{x-1}$ $g(x) = x^2$.

$$f\circ g(x) = f(g(x)) = \sqrt{g(x)-1} = \sqrt{x^2-1}$$

in order to make $f\circ g$ defined, we need $x^2-1 \geq 0$. i.e. $x^2 \geq 1$.
so the domain of $f\circ g$ is $(-\infty, -1] \cup [1, +\infty)$.

We also need to be careful when we try to reduce the expression of $f\circ g$:

Example. $f(x) = \sqrt{x}$, $g(x) = x^2$.

$$f\circ g(x) = \sqrt{g(x)} = \sqrt{x^2} = |x|$$

Next we will discuss about the manipulations on graphs:

Proposition. $c > 0$, a constant:

- If we shift graph of $y = f(x)$ upward by c units, we get the graph of $y = f(x) + c$.
- If we shift graph of $y = f(x)$ downward by c units, we get the graph of $y = f(x) - c$.
- If we shift graph of $y = f(x)$ to the left by c units, we get the graph of $y = f(x+c)$
- If we shift graph of $y = f(x)$ to the right by c units, we get the graph of $y = f(x-c)$

Proposition. $c>1$ is a constant:

- If we stretch the graph of $y=f(x)$ vertically by a factor of c, we get the graph of $y=cf(x)$.
- If we shrink the graph of $y=f(x)$ vertically by a factor of c, we get the graph of $y=\frac{1}{c}f(x)$.
- If we stretch the graph of $y=f(x)$ horizontally by a factor of c, we get the graph of $y=f(\frac{1}{c}x)$.
- If we shrink the graph of $y=f(x)$ horizontally by a factor of c, we get the graph of $y=f(cx)$
- If we reflect the graph of $y=f(x)$ about the x-axis, we get the graph of $y=-f(x)$.
- If we reflect the graph of $y=f(x)$ about the y-axis, we get the graph of $y=f(-x)$.
- If we reflect the graph of $y=f(x)$ about the origin, we get the graph of $y=-f(-x)$.

Example. We can obtain the graph of $g(x)=3x^2+2$ from that of $f(x)=x^2$ by first stretch vertically by a factor of 3 and the shift up vertically by 2 units.