Exponential Growth and Decay.

Consider $y(t)$ is the value of a quantity y at time t, and assume the rate of change of y with respect to t is proportional to its size $y(t)$ at at time. i.e. there is a constant k such that $\frac{dy}{dt} = ky$.

In many cases, the above assumption is quite reasonable.

For example, the growth of population, the compounded interest rate, and radioactive decay.

It turns out that we can find a function $y(t)$ satisfying $\frac{dy}{dt} = ky$, and it's an exponential function:

let $y(t) = Ce^{kt}$. then $\frac{dy}{dt} = Cke^{kt} = ky$.
and note that $y(0) = Ce^0 = C$, which means C is the initial value of $y(t)$.

Theorem. The only solutions of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions. $y(t) = y(0)e^{kt}$

Remark. If $k > 0$, we sometimes call $\frac{dy}{dt} = ky$ the law of natural growth.
If $k < 0$, we sometimes call $\frac{dy}{dt} = ky$ the law of natural decay.

Example (Population Growth).

$P(t)$ is the size of population at time t, which satisfies

$$\frac{dP}{dt} = kP \quad (k>0, \text{ constant})$$

Then $k = \frac{\frac{dP}{dt}}{P} = \frac{dP}{dt} \cdot \frac{1}{P}$ is called the relative growth rate. For a population growth, the relative growth rate remains a constant.

For example, if $\frac{dP}{dt} = 0.02P$, then the relative growth rate is 0.02, i.e. 2%, which means the population grows at a relative rate of 2% per year.

By our previous discussion on solutions to $\frac{dP}{dt} = kP$, we see if the population at time 0 is P_0, then $P(t) = P_0 e^{0.02t}$

Now let's think about a concrete example:
The world population was 2560 million in 1950, 3040 million in 1960. We try to use those data to recover the world population growth function:

let $P(t) = P(0)e^{kt}$. we let $t=0$ stand for the year 1950. then $t=10$ corresponds to the year 1960.

$P(t) = P(0)e^{kt}$, it follows $P(0) = 2560$.

$P(10) = 2560e^{10k} = 3040$.

So $k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017$, $\quad P(t) = 2560 e^{0.017t}$

We can use this to estimate population in $t=43$, i.e. the year 1993: $P(43) = 2560 e^{0.017 \times 43} \approx 5360$

Example (Radioactive Decay)

After studying the case of growth, we now study a situation where $k<0$, i.e. the exponential decay.

Radioactive substances decay by spontaneously emitting radiation, and mathematically, the decay can be described by its relative decay rate

$-\frac{1}{m}\frac{dm}{dt}$, where $m(t)$ is the mass at time t.

By some experiments, it has been found that the relative decay rate is a constant, so let $k=\frac{1}{m}\cdot\frac{dm}{dt}<0$, we can write

$$m(t)=m_0 e^{kt}$$

where $m_0=m(0)$ is its initial mass.

In physics, people describe the relative rate of decay by the term called half-life, which is the time required for half of any given quantity to decay:

$$\frac{m_0}{2}=m(T)=m_0 e^{kT} \Rightarrow kT=\ln\frac{1}{2} \Rightarrow T=\frac{1}{k}\ln\frac{1}{2}=-\frac{1}{k}\ln 2$$

Now let's compute an example: The half-life of Radium-226 is 1590 years. Let $m(t)$ be the mass of Ra after t years.

then $m(t)=m(0)e^{kt}$, so $\frac{m(0)}{2}=m(1590)=m(0)e^{k\cdot 1590}$.

we get $k=-\frac{\ln 2}{1590}$,

$$m(t)=m(0)\,e^{-\frac{\ln 2}{1590}t}=m(0)\left(e^{\ln 2}\right)^{-\frac{t}{1590}}=m(0)\cdot 2^{-\frac{t}{1590}}$$

If the initial mass of Ra is 100 mg, then after 1000 years, the mass is $m(1000)=100\cdot 2^{-\frac{1000}{1590}}\approx 65$ mg