As expected, f and f^{-1} are also related from the perspective of calculus.

Theorem. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem. If f is a one-to-one differentiable function with inverse function f^{-1}, and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Proof.

$$(f^{-1})'(a) = \lim_{x\to a} \frac{f^{-1}(x) - f^{-1}(a)}{x-a}, \quad \text{let } \begin{cases} y = f^{-1}(x) \\ b = f^{-1}(a) \end{cases}$$

$$= \lim_{y\to b} \frac{y-b}{f(y)-f(b)}$$

$$= \lim_{y\to b} \frac{1}{\frac{f(y)-f(b)}{y-b}}$$

$$= \frac{1}{\lim_{y\to b} \frac{f(y)-f(b)}{y-b}}$$

$$= \frac{1}{f'(b)}$$

$$= \frac{1}{f'(f(a))}$$

Remark. This can also be reflected by considering the graphs of f & f^{-1}. The corresponding slopes are inverses of each other.

Example. $f(x) = 2x + \cos x$ is a one-to-one function.

$f(0) = 1$. So $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{2}$

Now we can define the logarithmic function with base a, $f(x) = \log_a x$ to be the inverse function of the exponential function $y = a^x$.

i.e. We define $y = \log_a x \Leftrightarrow x = a^y$

Proposition: $\log_a(a^x) = x$ for every x in $(-\infty, +\infty)$

$a^{\log_a x} = x$ for every x in $(0, +\infty)$

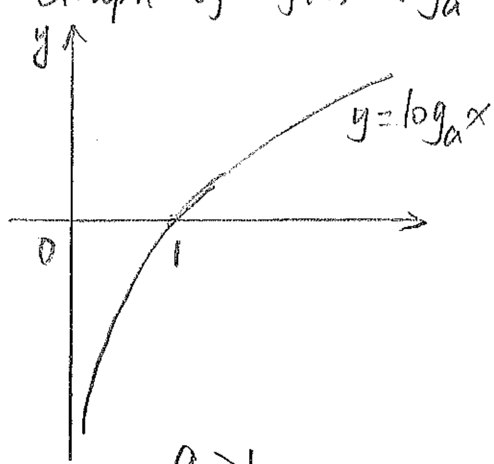
Properties: ① The domain of $f(x) = \log_a x$ is $(0, +\infty)$.
the image of $f(x) = \log_a x$ is $(-\infty, +\infty)$

② • $\log_a(xy) = \log_a x + \log_a y$ • $\log_a b = \dfrac{1}{\log_b a}$

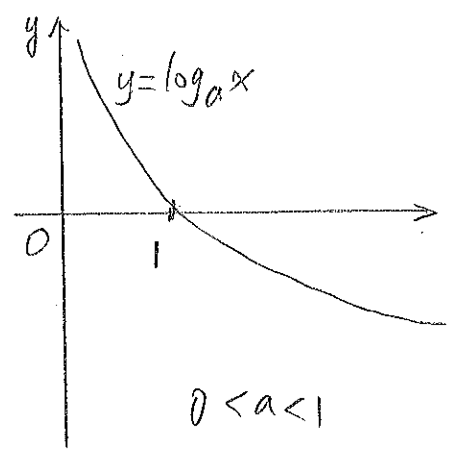
• $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$ • $\log_a 1 = 0$

• $\log_a x^r = r \log_a x$ • $\log_{a^s} b^r = \dfrac{r}{s} \log_a b$

③ Graph of $f(x) = \log_a x$:



$a > 1$



$0 < a < 1$

④: when $a>1$: $\lim_{x\to 0^+} \log_a x = -\infty$, $\lim_{x\to +\infty} \log_a x = +\infty$

when $0<a<1$: $\lim_{x\to 0^+} \log_a x = +\infty$, $\lim_{x\to +\infty} \log_a x = -\infty$

⑤ The y-axis is a vertical asymptote for $f(x)=\log_a x$

Example. Evaluate $\log_2 80 - \log_2 5$

$$\log_2 80 - \log_2 5 = \log_2 \frac{80}{5} = \log_2 16 = \log_2 2^4 = 4$$

Example. Find $\lim_{x\to 0} \log_{10}(\tan^2 x)$

As $x\to 0$, $\tan x \to 0$, $\tan^2 x \to 0^+$.

So $\log_{10}(\tan^2 x) \longrightarrow -\infty$

We conclude $\lim_{x\to 0} \log_{10}(\tan^2 x) = -\infty$

Recall that we've defined a real number e, which has the property that $(e^x)'=e^x$. Now we take e as the base for the logarithmic function $\log_e x$. and we usually denote it as $\ln x = \log_e x$, It's called the natural logarithm.

Corollary.
- $\ln e = 1$
- $e^{\ln x} = x$ for any $x>0$
- $\ln e^x = x$ for any x real number.

Example. Solve $e^{5-3x} = 10$:

$$\ln e^{5-3x} = \ln 10$$

$$5 - 3x = \ln 10$$

$$x = \frac{5 - \ln 10}{3}$$

Example. Write $a \ln x + b \ln y$ as a single logarithm:

$$a \ln x + b \ln y = \ln x^a + \ln y^b = \ln x^a y^b$$

Theorem. (Change of Base Formula) $a > 0$, $a \neq 1$. Then:

$$\log_a x = \frac{\ln x}{\ln a}$$

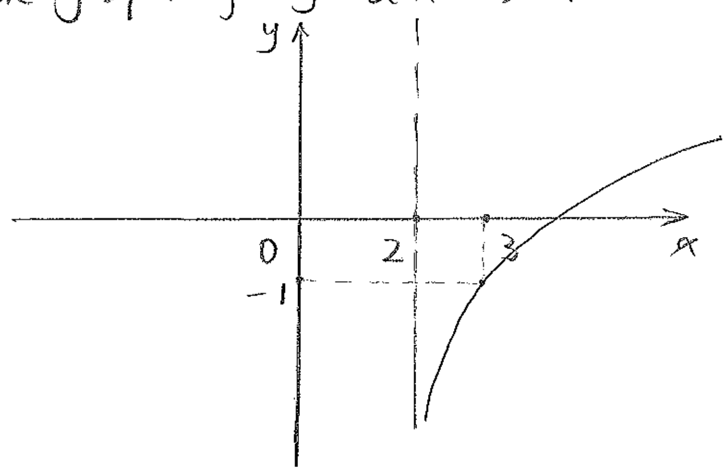
Proof. let $y = \log_a x$. then $x = a^y$

$$\frac{\ln x}{\ln a} = \frac{\ln a^y}{\ln a} = \frac{y \ln a}{\ln a} = y = \log_a x$$

Remark. There's a more general version of Change of Base Formula:

$$\log_a b = \frac{\log_c b}{\log_c a}$$ where a, b, c are positive numbers, $a \neq 1$, $c \neq 1$.

Example. Sketch the graph of $y = \ln(x-2) - 1$



Now we are going to discuss the derivatives of exponential and logarithmic functions:

First, we already showed that $(e^x)' = e^x$. The inverse function of $f(x) = e^x$ is $f^{-1}(x) = \ln x$, so we get

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

i.e. $(\ln x)' = \frac{1}{x}$.

Next, for any $a>0, a \neq 1$ constant,

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}\cdot(\ln x)' = \frac{1}{x \ln a}$$

So we obtain the following rule:

$(\log_a x)' = \frac{1}{x \ln a}$. In particular, $(\ln x)' = \frac{1}{x}$

Now we explore the derivative for $f(x) = a^x$:

$$f(x) = a^x = e^{\ln a^x} = e^{x \ln a}$$

so $(a^x)' = (e^{x \ln a})' = e^{x\ln a}\cdot \ln a = a^x \ln a$

We hence obtain the following rule:

$(a^x)' = a^x \ln a$. In particular, $(e^x)' = e^x$

Example. Find $\frac{d}{dx}\ln(\sin x)$.

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\cdot\frac{d}{dx}\sin x = \frac{\cos x}{\sin x} = \cot x$$

Example. Differentiate $f(x) = \ln \frac{x+1}{\sqrt{x-2}}$

$$\left(\ln \frac{x+1}{\sqrt{x-2}}\right)' = \left(\ln (x+1)(x-2)^{-\frac{1}{2}}\right)'$$
$$= \left(\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right)'$$
$$= \frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x-2}$$

Example. Find $f'(x)$ if $f(x) = \ln|x|$

$$f(x) = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0 \end{cases}$$

so when $x>0$. $f'(x) = \frac{1}{x}$.

when $x<0$. $f'(x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$

we conclude $f'(x) = \frac{1}{x}$. i.e. $(\ln|x|)' = \frac{1}{x}$

Example. Find $f'(x)$ if $f(x) = e^{\tan x}$

$$f'(x) = (e^{\tan x})' = e^{\tan x}\cdot\frac{1}{\cos^2 x}$$

There is an interesting application of the derivative $(\ln x)' = \frac{1}{x}$, which we call the Logarithmic Differentiation:

For some function $f(x)$, it is hard to find its derivative directly, but $\ln f(x)$ is much easier to differentiate sometimes so consider:

$$(\ln f(x))' = \frac{1}{f(x)}\cdot f'(x). \text{ We get } f'(x) = (\ln f(x))'\cdot f(x)$$

Example. Differentiate $y = x^x$.

$\ln x^x = x \ln x$

so $(x^x)' = x^x \cdot (x \ln x)' = x^x \cdot (x \cdot \frac{1}{x} + \ln x) = x^x (1 + \ln x)$

Example. Differentiate $y = \dfrac{\sqrt{x-3}}{(x+1)(x+2)}$

$$\ln y = \ln \frac{\sqrt{x-3}}{(x+1)(x+2)} = \ln (x-3)^{\frac{1}{2}}(x+1)^{-1}(x+2)^{-1}$$
$$= \frac{1}{2}\ln(x-3) - \ln(x+1) - \ln(x+2)$$

so $\dfrac{y'}{y} = \dfrac{1}{2} \cdot \dfrac{1}{x-3} - \dfrac{1}{x+1} - \dfrac{1}{x+2}$

$$y' = \frac{\sqrt{x-3}}{(x+1)(x+2)} \left[\frac{1}{2} \cdot \frac{1}{x-3} - \frac{1}{x+1} - \frac{1}{x+2} \right]$$

Example. Prove $(x^n)' = n x^{n-1}$ for $n \neq 0$.

$\ln x^n = n \ln x$

so $(x^n)' = x^n \cdot (n \ln x)' = x^n \cdot \frac{n}{x} = n x^{n-1}$