Exponential & Logarithmic Functions

Definition. An exponential function is one with the form $f(x) = a^x$, where $a > 0$, $a \neq 1$ is a constant.

Let's discuss the meaning of exponential:

When x is a positive integer, we know

$$f(x) = \underbrace{a \cdot a \cdot \cdots \cdot a}_{x \text{ copies}}$$

When $x = 0$, $f(0) = a^0 = 1$

When x is a negative integer, say $x = -n$.

$$f(x) = a^{-n} = \frac{1}{a^n}$$

When x is a rational number $x = \frac{p}{q}$, where p & q are integers and $q > 0$.

$$f(x) = a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

When x is an irrational number, we define:

$$f(x) = a^x = \lim_{r \to x} a^r, \quad r \text{ rational}.$$

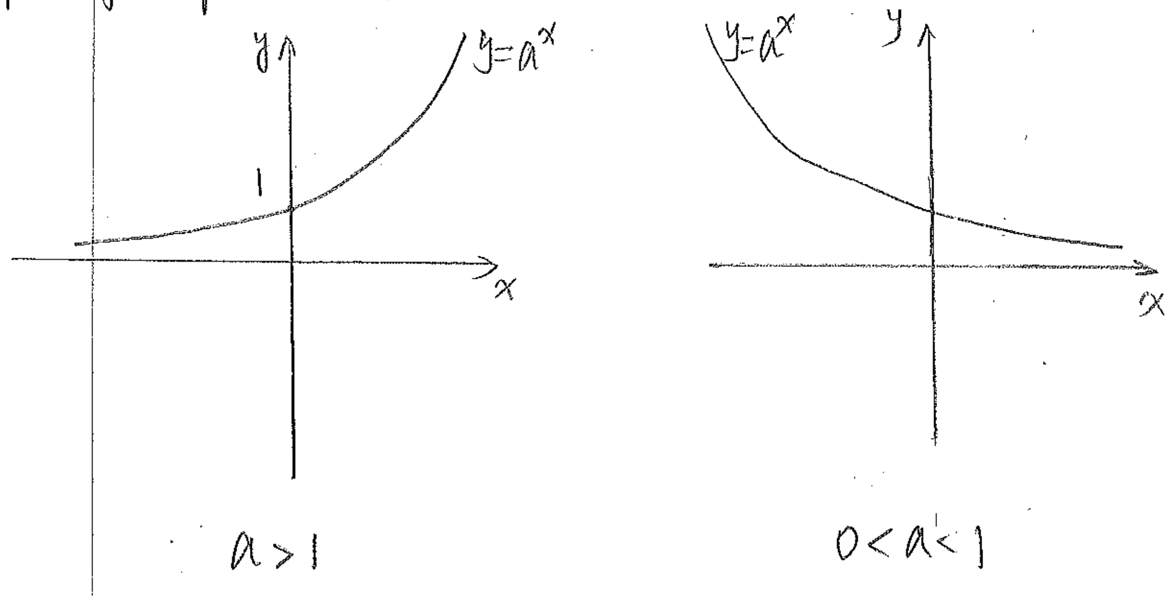
For example, how to define 2^π?

We take the sequence of rational numbers which converges to π: $3, 3.1, 3.14, 3.141, 3.1415, \ldots$

Then 2^π will be the limit of the following sequence:

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots$$

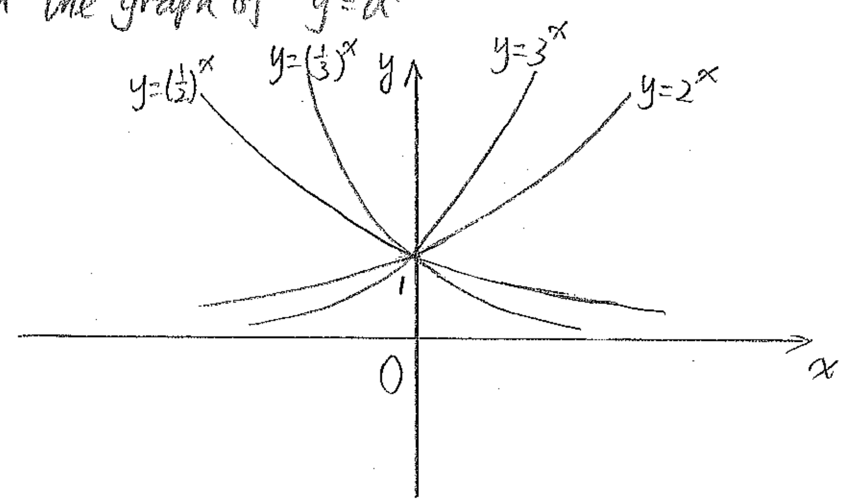
Graphs of exponential functions:



Observation: ① The graph of $y=a^x$ and the graph of $y=b^x$ are symmetric with respect to y-axis if and only if $ab=1$, i.e. $b=\frac{1}{a}$.

② If $1<a<b$, then the graph of $y=b^x$ is closer to y-axis than the graph of $y=a^x$.

③ If $0<b<a<1$, then the graph of $y=b^x$ is closer to y-axis than the graph of $y=a^x$



Theorem ① If $a>0$ and $a\neq 1$, then $f(x)=a^x$ is a continuous function with domain $(-\infty,+\infty)$ and range $(0,+\infty)$.

② If $a,b>0$, $x,y\in(-\infty,+\infty)$, then:

$$a^{x+y}=a^x\cdot a^y,\quad a^{x-y}=\frac{a^x}{a^y},\quad (a^x)^y=a^{xy},\quad (ab)^x=a^x b^x$$

③ If $a>1$, then

$$\lim_{x\to+\infty} a^x=+\infty,\quad \lim_{x\to-\infty} a^x=0$$

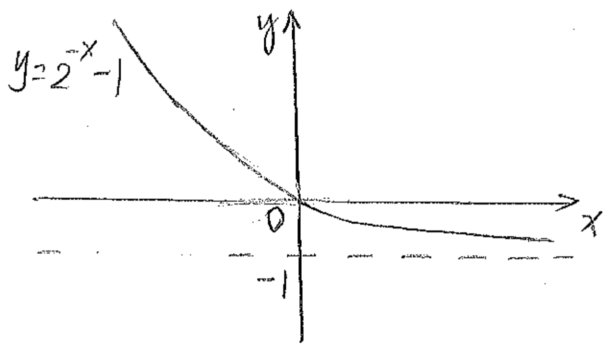
If $0<a<1$, then

$$\lim_{x\to+\infty} a^x=0,\quad \lim_{x\to-\infty} a^x=+\infty$$

We see in both cases, the x-axis is a horizontal asymptote.

Example. Find $\lim_{x\to\infty}(2^{-x}-1)$ and sketch $y=2^{-x}-1$.

$$\lim_{x\to\infty}(2^{-x}-1)=\lim_{x\to\infty}\left(\tfrac{1}{2}\right)^x-1=0-1=-1$$



If we try to compute the derivative of an exponential function, we will see the following:

$$(a^x)' = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h-1)}{h} = a^x \cdot \lim_{h\to 0} \frac{a^h-1}{h}$$

So $(a^x)'$ and a^x are proportional by a constant of

$$\lim_{h\to 0} \frac{a^h-1}{h}.$$

If for some choice of a, $\lim_{h\to 0} \frac{a^h-1}{h} = 1$, then for this a, $(a^x)'$ will be the same as a^x. In other words, we will obtain a function whose derivative is itself!

It turns out we can take the number $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$ to play this role:

when h is close to 0, $(1+h)^{\frac{1}{h}}$ is close to e,
so $1+h = [(1+h)^{\frac{1}{h}}]^h$ is close to e^h.

$$\lim_{h\to 0} \frac{e^h-1}{h} = \lim_{h\to 0} \frac{(1+h)-1}{h} = \lim_{h\to 0} 1 = 1$$

We call e the Euler's number.

Theorem. ① e is an irrational number
② $e = 2.71828\ldots$

Example. Evaluate $\lim_{x\to 0^-} e^{\frac{1}{x}}$.

let $y = \frac{1}{x}$, then $y \to -\infty$ as $x \to 0^-$.

$$\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{y\to -\infty} e^y = 0$$

Example. Find $\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1}$.

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{e^{2x}}} = \frac{1}{1+\lim_{x\to\infty}\frac{1}{e^{2x}}} = \frac{1}{1+0} = 1$$

Example. Find $\lim_{x\to\infty} \frac{e^{x}}{e^{2x}+1}$.

$$\lim_{x\to\infty} \frac{e^{x}}{e^{2x}+1} = \lim_{x\to\infty} \frac{1}{e^{x}+\frac{1}{e^{x}}} = 0$$

We see in our previous discussion that $(a^x)' = a^x \lim_{h\to 0} \frac{a^h-1}{h}$, and we have defined the number e such that $(e^x)' = e^x$, but it still remains a question now how to find the value $\lim_{h\to 0} \frac{a^h-1}{h}$ for any given $a>0$ and $a\neq 1$.

It turns out that we need to first study the logarithmic function and then come back to this point. We will study it in a more general approach, that is, discussing in general about inverse functions, and apply the discussion to exponential functions and logarithmic functions.

Definition. We say a function f is one-to-one (more formally, it's also called injective) if for any $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Example. $f(x) = x^2$ is NOT one-to-one,
$f(x) = x^3$ is one-to-one.

Theorem. (Horizontal Line Test). A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Definition. Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A, and is defined by $f^{-1}(y) = x \Leftrightarrow f(x) = y$ for any y in B.

We see from the definition that:

domain of f^{-1} is same as range of f.
range of f^{-1} is same as domain of f.

Example. The inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$:

if $y = f(x) = x^3$, then $x = y^{\frac{1}{3}}$.

i.e. $x = f^{-1}(y) = y^{\frac{1}{3}}$.

Theorem. (Cancellation Law) If f^{-1} is the inverse function of f,

then:
- $f^{-1}(f(x)) = x$ for every x in A
- $f(f^{-1}(x)) = x$ for every x in B.

Example. Again take $f(x) = x^3$. we know $f^{-1}(x) = x^{\frac{1}{3}}$.

$f^{-1}(f(x)) = (x^3)^{\frac{1}{3}} = x$. $\quad f(f^{-1}(x)) = (x^{\frac{1}{3}})^3 = x$.

In general, we can find the inverse of a given one-to-one function in the following way: Based on the expression of $y = f(x)$, try to write x in terms of y: $x = f^{-1}(y)$.
At last, switch x & y in the expression $x = f^{-1}(y)$

Example. Find the inverse function of $f(x) = x^3 + 2$.

Let $y = f(x) = x^3 + 2$.

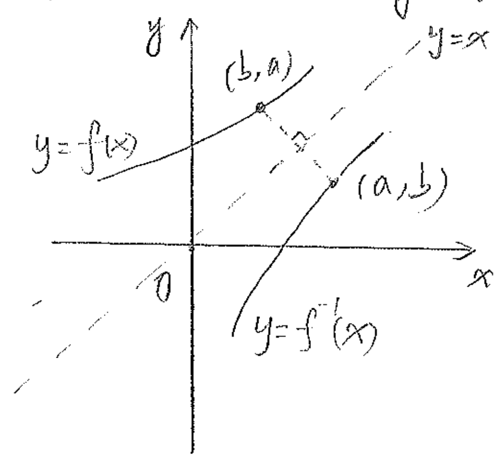
then $x^3 = y - 2 \Rightarrow x = \sqrt[3]{y-2}$.

so $x = f^{-1}(y) = \sqrt[3]{y-2}$.

Interchange x & y we get

$$y = f^{-1}(x) = \sqrt[3]{x-2}$$

Proposition. The graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$



Example. Sketch the graph of $y = \sqrt{x}$ based on the graph of $y = x^2$ $(x \geq 0)$

If $f(x) = x^2$ with domain given by $x \geq 0$.
then f is one-to-one and has inverse function
$f^{-1}(x) = \sqrt{x}$.