# IMPLICIT DIFFERENTIATION

A variable $y$ is a function of a variable $x$ in general does not always imply we can find easily an expression of the form $y=f(x)$.

Instead, the relation between $x$ & $y$ may be given by an equation of the form $F(x,y)=0$. In such cases, we still can compute the derivative $y'$ by the method of implicit differentiation.

For example, $y^3+3x^2y=13$ defines a function $y=f(x)$ in an implicit way. It is hard to compute the explicit expression $y=f(x)$, but we can find $f'(x)$ in the following way:

Implicit Differentiation:

Step 1. Differentiate each side of the equation with respect to $x$, viewing $y$ as a function of $x$

Step 2. Solve for $y'$.

So by this method, for the question above, we differentiate both sides of $y^3+3x^2y=13$:

$$3y^2\cdot y' + 3(2x\cdot y + x^2y') = 0$$

$$\Rightarrow\ y^2\cdot y' + 2xy + x^2y' = 0$$

$$\Rightarrow\ y' = \frac{-2xy}{x^2+y^2}$$

Note that the expression of $y'$ obtained from Implicit Differentiation often contain both $x$ & $y$.

Example. (i) If $x^2+y^2=25$, Find $\frac{dy}{dx}$

(ii) Find an equation of the tangent line to the circle $x^2+y^2=25$ at $(3,4)$.

(i). Differentiate both sides of $x^2+y^2=25$ with respect to $x$:

$$2x+2y\cdot y'=0 \quad \text{so} \quad y'=-\frac{x}{y}$$

(ii) at $(3,4)$, $y'=-\frac{3}{4}$ so the slope of the tangent line is $-\frac{3}{4}$.

the equation of the tangent line is

$$y-4=-\frac{3}{4}(x-3)$$

Example. Find $y'$ if $\sin(x+y)=y^2\cos x$

Differentiate both sides with respect to $x$:

$$\cos(x+y)\cdot(1+y')=2y\cdot y'\cos x-y^2\sin x$$

$$y'=\frac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)}$$

Example. Find $y''$ if $x^4+y^4=16$

$$4x^3+4y^3\cdot y'=0$$

we get $y'=-\frac{x^3}{y^3}$.

$$y''=-\frac{3x^2\cdot y^3-3y^2\cdot y'x^3}{y^6}=-\frac{3x^2y^3+3y^2\cdot\frac{x^3}{y^3}\cdot x^3}{y^6}$$

$$=-\frac{3x^2y^4+3x^6}{y^7}$$

$$=-\frac{3x^2(y^4+x^4)}{y^7}=-\frac{48x^2}{y^7}$$

# LINEAR APPROXIMATIONS AND DIFFERENTIALS

Recall that $f'(b) = \lim_{h \to 0} \frac{f(b+h)-f(b)}{h}$

It means when $h$ is close to 0. $f'(b) \approx \frac{f(b+h)-f(b)}{h}$

i.e. $f(b+h)-f(b) \approx f'(b)\,h$

$$f(b+h) \approx f(b) + f'(b)\,h$$

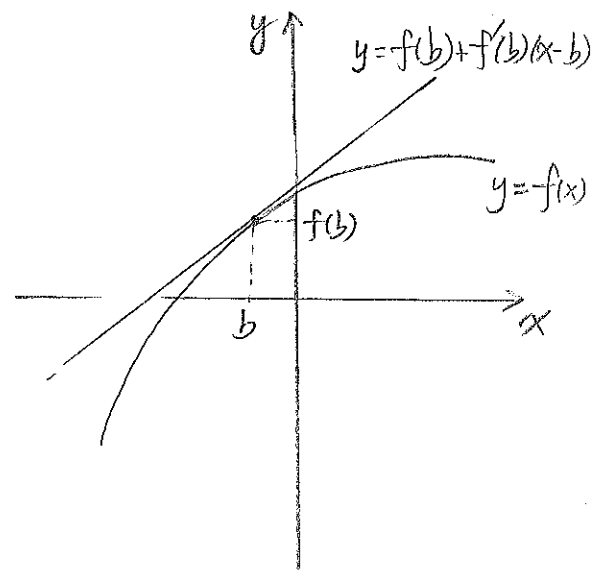
Now if we write $x = b+h$, then for $x$ close to $b$:

$$f(x) \approx f(b) + f'(b)(x-b)$$

We see $f(x)$ is approximated by the linear function

$$L(x) = f(b) + f'(b)(x-b)$$

But this is the equation of the tangent line of $f(x)$ at $(b, f(b))$. We therefore find that the value of a differentiable function $f$ near $b$ can be approximated by its tangent line at $(b, f(b))$. This approximation is called the linear approximation at $b$, and $L(x)$ is called the linearization of $f$ at $b$.



Example. Find the linear approximation of $f(x)=\sqrt{x}$ at $x=1$.

$f'(x)=\frac{1}{2}\frac{1}{\sqrt{x}}$. So $f(1)=1$ and $f'(1)=\frac{1}{2}$.

near $x=1$. $f(x)\approx f(1)+f'(1)(x-1)=1+\frac{1}{2}(x-1)$

$f(1.01)\approx 1+\frac{1}{2}(1.01-1)=1+\frac{1}{2}\times 0.01=1.005$

If you check by using a computer. $f(1.01)=1.00498756\ldots$

Example. $f(x)=\sin x$. $f'(x)=\cos x$. $f(0)=0$, $f'(0)=1$.

so when $x$ is close to $0$:

$$f(x)\approx f(0)+f'(0)(x-0)=x$$

Next we introduce the concept of differential:

$f$ is a differentiable function. Denote $dx$ to be the change of $x$. Define $dy=df=f'(x)dx$ to be the differential of the function $y=f(x)$.

When $x$ changes by $dx$, the corresponding change in the value of the function $f$ is:

$$f(x+dx)-f(x).$$

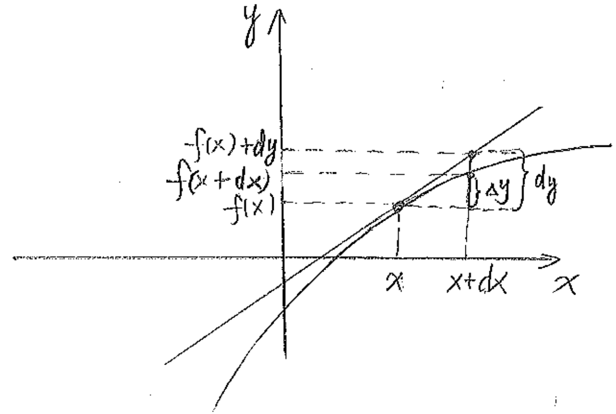
If $dx$ is a small amount, then linear approximation tells us

$$f(x+dx)\approx f(x)+f'(x)\cdot dx$$

$$\text{i.e. } f(x+dx)-f(x)\approx f'(x)dx=df.$$

So $df$ is the approximation of the change of function by linear approximation when $x$ changes by $dx$.

The meaning of $dy$ can be illustrated by the following graph.




Example. If $f(x) = \sqrt{x}$. $f'(x) = \frac{1}{2\sqrt{x}}$.

So $dy = f'(x)dx = \frac{1}{2\sqrt{x}}\, dx$

As we've computed in the previous example.
when $x=1$, $dx = 0.01$. we see

$$dy = \frac{1}{2\times\sqrt{1}} \times 0.01 = 0.005$$

so $\Delta y = f(x+dx) - f(x) = f(1+0.01) - f(1) \approx dy = 0.005$

Example. The radius of a sphere was measured to be 21 cm, with a possible error of at most 0.05 cm. What's the maximum error in using this value of radius to compute the volume of the sphere?

We know the volume of a sphere is $V = \frac{4}{3}\pi r^3$
denote the error of measurement by $dr$. then

$$dV = V' \cdot dr = 4\pi r^2 dr$$

When $r=21$, $dr = 0.05$, $dV = 4\pi \cdot 21^2 \times 0.05 \approx 277$.
so the maximum error is about $277\ cm^3$.

The relative error $\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = \frac{3}{r} dr = \frac{3}{21} \times 0.05$

$\approx 0.7\%$