THE CHAIN RULE

Theorem (The Chain Rule) If f, g are both differentiable and $F = f \circ g$ is the composition function: $F(x) = f(g(x))$, then F is differentiable and F' is given by:

$$F'(x) = f'(g(x)) \cdot g'(x).$$

If we write $y = f(u)$ and $u = g(x)$, the above theorem can be written as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

We can write $F(x)$ as a composition:

$F(x) = f(g(x))$, where $f(u) = \sqrt{u}$ and $u = g(x) = x^2+1$

So by the Chain Rule,

$$\frac{dF}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \cdot 2x = \frac{1}{2} \frac{1}{\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Example $y = \sin(x)^2$.

We can decompose y as: $y = \sin u$, $u = x^2$.

by the Chain Rule. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u) \cdot 2x = 2x \cos x^2$

Example $y = \sin^2 x$

We can decompose y as: $y = u^2$, $u = \sin x$

by the Chain Rule. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \cos x = 2 \sin x \cos x$

A special case of the Chain Rule that is used quite often is the following:

Example. $g(x)$ is differentiable. the derivative of $y = g(x)^n$ is

$$\frac{dy}{dx} = n\, g(x)^{n-1} \cdot g'(x)$$

Example. Differentiate $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$

$f(x) = (x^2+x+1)^{-\frac{1}{3}}$.

So $f'(x) = -\frac{1}{3}(x^2+x+1)^{-\frac{1}{3}-1} \cdot (x^2+x+1)'$

$= -\frac{1}{3}(x^2+x+1)^{-\frac{4}{3}} \cdot (2x+1)$

Example. Differentiate $f(x) = \left(\frac{x-2}{2x+1}\right)^9$.

$f'(x) = 9\left(\frac{x-2}{2x+1}\right)^8 \cdot \left(\frac{x-2}{2x+1}\right)'$

$= 9 \cdot \frac{(x-2)^8}{(2x+1)^8} \cdot \frac{1\cdot(2x+1) - 2\cdot(x-2)}{(2x+1)^2}$

$= 9 \cdot \frac{(x-2)^8}{(2x+1)^8} \cdot \frac{5}{(2x+1)^2}$

$= \frac{45(x-2)^8}{(2x+1)^{10}}$

Example. Differentiate $f(x) = (2x+1)^5(x^3-x+1)^4$

$f'(x) = [(2x+1)^5]'(x^3-x+1)^4 + (2x+1)^5[(x^3-x+1)^4]'$

$= 5(2x+1)^4 \cdot (2x+1)'(x^3-x+1)^4 + (2x+1)^5 \cdot 4(x^3-x+1)^3 \cdot (x^3-x+1)'$

$= 5(2x+1)^4 \cdot 2 \cdot (x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3 \cdot (3x-1)$

$= 10(2x+1)^4(x^3-x+1)^4 + 4(3x-1)(2x+1)^5(x^3-x+1)^3$

More general, we can compute the derivative of composition of more than two functions by repeatedly apply the chain Rule:

If $y=f(u)$, $u=g(v)$ and $v=h(x)$,
then the derivative of $y=f(g(h(x)))$ is.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

Example. Differentiate $f(x)=(\sin^2 x+\sin x)^4$

We can let $y=f(x)$, then $f(x)$ can be decomposed into
$y=u^4$, $u=v^2+v$, $v=\sin x$.

so
$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}\\&=4u^3\cdot(2v+1)\cdot\cos x\\&=4(\sin^2 x+\sin x)^3\cdot(2\sin x+1)\cdot\cos x\end{aligned}$$

Example. Differentiate $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$

$$y=\left(x+\left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

so
$$\begin{aligned}y'&=\frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\cdot\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)'\\&=\frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\cdot\left(1+\left[(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right]'\right)\\&=\frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}\cdot(x+x^{\frac{1}{2}})^{-\frac{1}{2}}\cdot(x+x^{\frac{1}{2}})'\right)\\&=\frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}(x+x^{\frac{1}{2}})^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)\right)\end{aligned}$$