Example. The position of a particle is $s = f(t) = t^3 - 6t^2 + 9t$.

(i) Find the intervals the particle is moving forward and the intervals the particle is moving backward.

(ii). Find the total distance traveled by the particle during the first 5 seconds.

(iii). Find the acceleration function $a(t)$.

(iv) Sketch the graph of position, velocity and acceleration for the first 5 seconds.

(i) First, we need to find the velocity function:

$$v(t) = \frac{ds}{dt} = f'(t) = (t^3 - 6t^2 + 9t)' = 3t^2 - 12t + 9 = 3(t-1)(t-3)$$

we know the particle is moving forward if $v(t) > 0$ and moving backward if $v(t) < 0$.

$v(t) = 3(t-1)(t-3)$. So $v(t) > 0$ when $t < 1$ or $t > 3$.
$v(t) < 0$ when $1 < t < 3$.

(ii). We need to compute the distance in two cases: moving forward and moving backward, and then add them up.

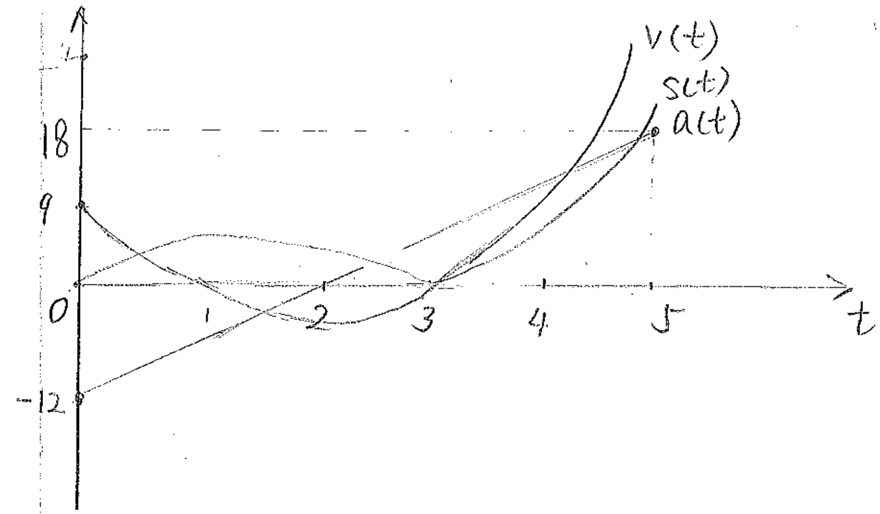
On $(0, 1)$: $|f(1) - f(0)| = |4 - 0| = 4$.

On $(3, 5)$: $|f(5) - f(3)| = |20 - 0| = 20$

On $(1, 3)$: $|f(3) - f(1)| = |0 - 4| = 4$

So the total distance travelled is $4 + 20 + 4 = 28$.

(iii) $a(t) = v'(t) = (3t^2 - 12t + 9)' = 6t - 12$



Example. $C(x)$ is the total cost for producing x units of commodity. We call $C(x)$ the cost function. If the production is increased to x_2 from x_1, the additional cost is $C(x_2)-C(x_1)$, and the average rate of change is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2)-C(x_1)}{x_2-x_1} = \frac{C(x_1+\Delta x)-C(x_1)}{\Delta x}$$

Taking the limit of the average rate of change as $\Delta x \to 0$, we obtain the instantaneous rate of change:

$$\lim_{\Delta x \to 0} \frac{C(x_1+\Delta x)-C(x_1)}{\Delta x} = C'(x_1)$$

we call it the marginal cost, which has the following interpretation:

when $x_1 = n$ is big, and $\Delta x = 1$.

Since $\displaystyle\lim_{\Delta x \to 0} \frac{C(n+\Delta x)-C(n)}{\Delta x} = C'(n)$

it follows $\dfrac{C(n+1)-C(n)}{1} \approx C'(n)$. (Compared to n, 1 is already very close to 0).

i.e. $C'(n)$, the marginal cost at n, approximates the increase in cost if one more unit is produced when the original production is n units.

Example. A company has the cost function $C(x) = 10000 + 5x + 0.01x^2$ when x units of goods is produced.

We get the marginal cost is $C'(x) = 5 + 0.02x$
it tells us

$$C(501) - C(500) \approx C'(500) = 5 + 0.02 \times 500 = 15.$$

$$\text{so } C(501) \approx C(500) + 15 = 10000 + 5 \times 500 + 0.01 \times 500^2 + 15$$
$$= 10000 + 2500 + 2500 + 15$$
$$= 15015$$

And the actual
$$C(501) = 10000 + 5 \times 501 + 0.01 \times 501^2$$
$$= 10000 + 2505 + 2510.01$$
$$= 15015.01$$

We see the marginal cost provides a good estimation of the actual increase.

PRODUCT RULE AND QUOTIENT RULE

- The Product Rule: If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

i.e. $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$

- The Quotient Rule: If f and g are both differentiable, $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$

i.e. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

We are going to see why the rules are true:

If f and g are differentiable.

$$\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x)}{h} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

so we have proved the product rule.

Next we are going to show the quotient rule:

If f and g are differentiable with $g(x) \neq 0$,

$$\left(\frac{1}{g(x)}\right)' = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h \to 0} \frac{g(x) - g(x+h)}{g(x+h)g(x)} \cdot \frac{1}{h}$$

$$= -\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \to 0} \frac{1}{g(x+h)g(x)}$$

$$= -g'(x) \cdot \frac{1}{g(x)^2}$$

$$= -\frac{g'(x)}{g(x)^2}$$

so $$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x) \cdot \frac{1}{g(x)}\right)' = f'(x) \cdot \frac{1}{g(x)} + \left(\frac{1}{g(x)}\right)' f(x)$$

$$= f'(x) \cdot \frac{1}{g(x)} - \frac{g'(x)}{g(x)^2} \cdot f(x)$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

We proved the quotient rule.

Example. $f(x) = x^2 \sin x$

$$f'(x) = (x^2 \sin x)' = (x^2)' \sin x + x^2 (\sin x)'$$

$$= 2x \sin x + x^2 \cos x$$

Example. If $h(x) = x g(x)$, and $g(3) = 5$, $g'(3) = 2$. find $h'(3)$

$$h'(x) = (x g(x))' = (x)' g(x) + x \cdot (g(x))' = g(x) + x g'(x)$$

so $h'(3) = g(3) + 3g'(3) = 5 + 3 \times 2 = 11$

Example. $f(x) = \dfrac{x^2+x-2}{x^3+6}$

$$f'(x) = \frac{(x^2+x-2)'(x^3+6) - (x^3+6)'(x^2+x-2)}{(x^3+6)^2}$$

$$= \frac{(2x+1)(x^3+6) - (3x^2)(x^2+x-2)}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

We can use the quotient rule to obtain the derivative of the tangent function $f(x) = \tan x$:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - (-\sin x)\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Similarly, we can obtain the derivative of the other trigonometric functions:

$$(\cot x)' = -\csc^2 x, \quad (\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x$$