Example. The position of a particle is \( s = f(t) = t^3 - 6t^2 + 9t \).

(i) Find the intervals the particle is moving forward and the intervals the particle is moving backward.

(ii) Find the total distance traveled by the particle during the first 5 seconds.

(iii) Find the acceleration function \( a(t) \).

(iv) Sketch the graph of position, velocity, and acceleration for the first 5 seconds.

(i) First, we need to find the velocity function:

\[
V(t) = \frac{ds}{dt} = f'(t) = (t^3 - 6t^2 + 9t)' = 3t^2 - 12t + 9 = 3(t-1)(t-3)
\]

We know the particle is moving forward if \( V(t) > 0 \) and moving backward if \( V(t) < 0 \).

\[
V(t) = 3(t-1)(t-3).
\]

So \( V(t) > 0 \) when \( t < 1 \) or \( t > 3 \).

\( V(t) < 0 \) when \( 1 < t < 3 \).

(ii) We need to compute the distance in two cases: moving forward and moving backward, and then add them up.

On \((0, 1)\):

\( |f(1) - f(0)| = 14 - 0 = 14 \)

On \((3, 5)\):

\( |f(5) - f(3)| = 20 - 0 = 20 \)

On \((1, 3)\):

\( |f(3) - f(1)| = 10 - 4 = 6 \)

So the total distance traveled is \( 14 + 20 + 6 = 40 \).

(iii) \( a(t) = V'(t) = (3t^2 - 12t + 9)' = 6t - 12 \).
Example. $C(x)$ is the total cost for producing $x$ units of commodity. We call $C(x)$ the cost function. If the production is increased from $x_1$ to $x_2$, the additional cost is $C(x_2) - C(x_1)$, and the average rate of change is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Taking the limit of the average rate of change as $\Delta x \to 0$, we obtain the instantaneous rate of change:

$$\lim_{\Delta x \to 0} \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x} = C'(x_1)$$

we call it the marginal cost, which has the following interpretation:

when $x_1 = n$ is big, and $\Delta x = 1$.

Since $\lim_{\Delta x \to 0} \frac{C(n+\Delta x) - C(n)}{\Delta x} = C'(n)$

it follows $\frac{C(n+1) - C(n)}{1} \approx C'(n)$, (compared to $n$, $1$ is already very close to 0).

1.e. $C'(n)$, the marginal cost at $n$, approximates the increase in cost if one more unit is produced when the original production is $n$ units.
Example: A company has the cost function \( C(x) = 10000 + 5x + 0.01x^2 \) when \( x \) units of goods is produced.

We get the marginal cost is \( C'(x) = 5 + 0.02x \) it tells us

\[
C(501) - C(500) \approx C'(500) = 5 + 0.02 \times 500 = 15
\]

So \( C(501) \approx C(500) + 15 = 10000 + 5 \times 500 + 0.01 \times 500^2 + 15 \)

\[
= 10000 + 2500 + 2500 + 15
\]

\[
= 15015
\]

And the actual \( C(501) = 10000 + 5 \times 501 + 0.01 \times 501^2 \)

\[
= 10000 + 2505 + 2510.01
\]

\[
= 15155.01
\]

We see the marginal cost provides a good estimation of the actual increase.
PRODUCT RULE AND QUOTIENT RULE

The Product Rule: If $f$ and $g$ are both differentiable, then
\[
\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]
\]
i.e. \( (f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \)

The Quotient Rule: If $f$ and $g$ are both differentiable, \( g(x) \neq 0 \),
then
\[
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{(g(x))^2}
\]
i.e. \( \left(\frac{f(x)}{g(x)}\right)' = \frac{f(x)g'(x) - g(x)f'(x)}{(g(x))^2} \)

We are going to see why the rules are true:

If $f$ and $g$ are differentiable,
\[
(f(x)g(x))' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}
\]
\[
= \lim_{h \to 0} \left[ \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \right]
\]
\[
= \lim_{h \to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} g(x)
\]
\[
= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
\[
= f(x)g'(x) + g(x)f'(x)
\]

so we have proved the product rule.
Next we are going to show the quotient rule:

If \( f \) and \( g \) are differentiable with \( g(x) \neq 0 \),

\[
\frac{d}{dx} \left( \frac{1}{g(x)} \right) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{g(x) - g(x+h)}{g(x+h)g(x)} \cdot \frac{1}{h} \\
= -\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \\
= -\frac{g'(x)}{g(x)^2}
\]

So:

\[
\left( \frac{f(x)}{g(x)} \right)' = \left( f(x) \cdot \frac{1}{g(x)} \right)' = f'(x) \cdot \frac{1}{g(x)} + \left( \frac{1}{g(x)} \right)' f(x) \\
= f'(x) \cdot \frac{1}{g(x)} - \frac{g'(x)}{g(x)^2} \cdot f(x) \\
= \frac{f(x)g'(x) - g(x)f'(x)}{g(x)^2}
\]

We proved the quotient rule.

Example. \( f(x) = x^2 \sin x \)

\[
f'(x) = (x^2 \sin x)' = (x^2)' \sin x + x^2 (\sin x)' \\
= 2x \sin x + x^2 \cos x
\]

Example. If \( h(x) = x g(x) \) and \( g(3) = 5 \), \( g'(3) = 2 \), find \( h'(3) \)

\[
h'(x) = (x g(x))' = (x)' g(x) + x \cdot (g(x))' = g(x) + x g'(x)
\]

so \( h'(3) = g(3) + 3g'(3) = 5 + 3 \cdot 2 = 11 \)
Example. \( f(x) = \frac{x^2 + x - 2}{x^2 + 6} \)

\[
f'(x) = \frac{(x^2 + x - 2)'(x^2 + 6) - (x^2 + 6)'(x^2 + x - 2)}{(x^2 + 6)^2}
\]

\[
= \frac{(2x + 1)(x^2 + 6) - (3x^2)(x^2 + x - 2)}{(x^2 + 6)^2}
\]

\[
= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^2 + 6)^2}
\]

We can use the quotient rule to obtain the derivative of the tangent function \( f(x) = \tan x \):

\[
(tan x)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x}
\]

\[
= \frac{\cos x \cos x - (-\sin x) \sin x}{\cos^2 x}
\]

\[
= \frac{1}{\cos^2 x}
\]

\[
= \sec^2 x
\]

Similarly, we can obtain the derivative of the other trigonometric functions:

\( (\cot x)' = -\csc^2 x \), \( (\sec x)' = \sec x \tan x \), \( (\csc x)' = -\csc x \cot x \)