BASIC DIFFERENTIATION FORMULAS

We are going to discuss some formulas for the differentiation of functions of certain kinds.

- If $f(x) \equiv c$ is a constant function, then

$$\frac{d}{dx} f(x) = \frac{d}{dx} c = 0$$

The reason is not hard to see:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

- If $f(x) = x^n$ is a power function, where n is a real number such that $n \neq 0$, then

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^n) = n x^{n-1}$$

Example. ① $f(x) = \frac{1}{x^2}$, then $f'(x) = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3}$

② $f(x) = \sqrt[3]{x^2}$, then $f'(x) = \frac{d}{dx}(\sqrt[3]{x^2}) = \frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}}$

- General Rules for Differentiation:

The Constant Multiple Rule: If c is a constant and f is a differentiable function, then:

$$\frac{d}{dx}[c f(x)] = c \frac{d}{dx} f(x)$$

- The Sum Rule: If f and g are both differentiable, then:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

The proofs of both rules are straight forward:

If $g(x) = cf(x)$, then

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$
$$= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
$$= cf'(x)$$

If $k(x) = f(x) + g(x)$. then

$$k'(x) = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

With those rules we have discussed, we can take the derivative of any polynomial.

Example. $f(x) = x^8 - 5x^5 + 2x$

$$f'(x) = (x^8)' - (5x^5)' + (2x)'$$
$$= 8x^7 - 5(x^5)' + 2(x)'$$
$$= 8x^7 - 5\times 5x^4 + 2\times 1$$
$$= 8x^7 - 25x^4 + 2$$

Example. $f(x) = \frac{3}{x} - 2x^2$

$$f'(x) = \left(\tfrac{3}{x}\right)' - (2x^2)' = 3\cdot(x^{-1})' - 2(x^2)'$$
$$= 3\cdot(-x^{-2}) - 2\cdot(2x)$$
$$= -\frac{3}{x^2} - 4x$$

Next we are going to find the derivative of trigonometric functions :

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$

The proof is based on the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$, $\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}$$

$$= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x$$

Similarly.

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}$$

$$= -\sin x$$

Example. $f(x) = 3\sin x + 4\cos x$

$f'(x) = 3(\sin x)' + 4(\cos x)' = 3\cos x - 4\sin x$

Example. Find the 27-th derivative of $f(x) = \cos x$

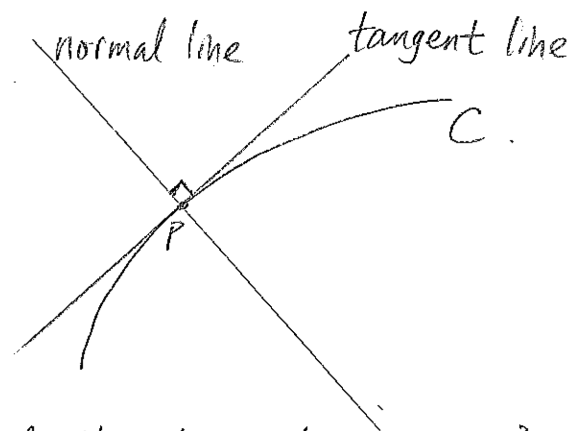
$f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f''''(x) = \cos x$

So we see if we take derivative of $f(x)$ four times. the result turns out to be back to $f(x)$

So $f^{(24)}(x) = \cos x$, $f^{(25)} = -\sin x$, $f^{(26)} = -\cos x$, $f^{(27)} = \sin x$

Definition. The normal line to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

Proposition. The slope of the tangent line and the slope of the normal line to a curve C at a point P are negative reciprocal of each other.



Example. Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at $(1, 1)$.

$f'(x) = (x\sqrt{x})' = (x^{\frac{3}{2}})' = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$f'(1) = \frac{3}{2}\sqrt{1} = \frac{3}{2}$. So the slope of the tanged line is $\frac{3}{2}$.
The equation of the tangent line at $(1,1)$ is $y - 1 = \frac{3}{2}(x-1)$.

The slope of the normal line is $-\frac{1}{\frac{3}{2}} = -\frac{2}{3}$,
The equation of the normal line at $(1,1)$ is $y - 1 = -\frac{2}{3}(x-1)$