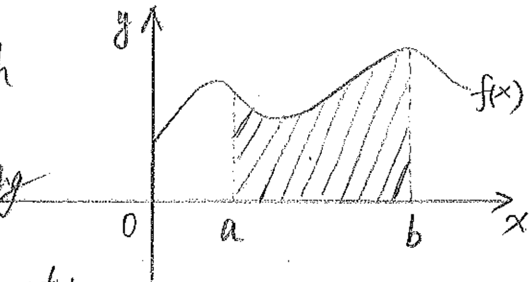
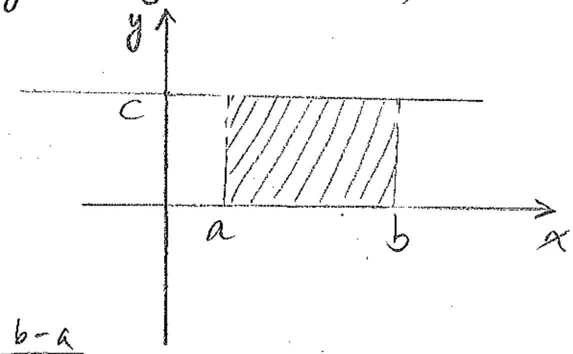
AREAS AND DISTANCES

We are going to define the concept of area for a region bounded between the curve of $f(x)$ and x-axis on an interval $[a, b]$.



It seems not easy to come up with a general formula for defining the area, but we can start by thinking about what we know:
If the function is a constant function $f(x) \equiv C$ on $[a, b]$, then we know what the area is: just the area of the corresponding rectangle $C \cdot (b-a)$

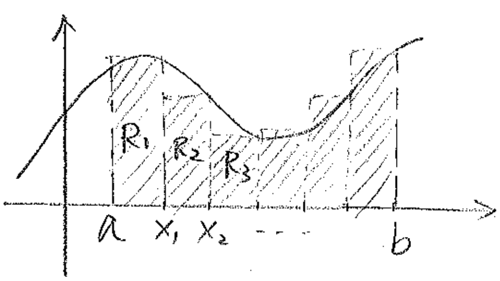


This motivates us the following construction:

Given a natural number n, we divide $[a, b]$ into n pieces, each of which has length $\Delta x := \frac{b-a}{n}$

So the endpoints of the small pieces are

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, \quad x_n = a + n \cdot \Delta x = b$$



Now consider the rectangles with base $[x_i, x_{i+1}]$ and height $f(x_i)$ for all $1 \leq i \leq n$. We sum up the areas of these rectangles:

$$A = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$
$$= [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

As $n \to +\infty$, we see the region covered by these small rectangles converges to the region between the graph of $y = f(x)$ and x-axis.

So we define the area A of the region that lies under the graph of a continuous function f and above x-axis on the interval $[a, b]$ is the limit:

$$A = \lim_{n\to+\infty} R_n = \lim_{n\to+\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

It can be proved that if we consider the rectangles above $[x_i, x_{i+1}]$ with height $f(x_{i-1})$ instead of $f(x_i)$, then going through the above procedures will lead to the same result, i.e.

$$A = \lim_{n\to+\infty} L_n = \lim_{n\to+\infty} [f(x_0) + f(x_1) + \cdots + f(x_{n-1})]\Delta x$$

More generally, we can also take arbitrary point $x_i^* \in [x_i, x_{i+1}]$ and consider the rectanges above $[x_i, x_{i+1}]$ with height $f(x_i^*)$. We then have

$$A = \lim_{n\to+\infty} [f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]\Delta x$$

Now we introduce a new notation to sum up a sequence: If $a_1, a_2, \ldots, a_n$ is a sequence, we'll denote their sum by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

So $A = \lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} f(x_{i-1})\Delta x = \lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

Example. Compute the area A of the region that lies below $f(x) = x^2$, above x-axis, and on the interval $[0, 1]$, using both R_n and L_n methods. We need to make use of an algebraic formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$, so $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, $\ldots$, $x_n = \frac{n}{n} = 1$.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \frac{\sum_{i=1}^{n} i^2}{n^3} = \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{(n+1)(2n+1)}{6n^2}$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{x\to\infty} \frac{1}{6}\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{2n+1}{n}\right) = \lim_{x\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}$$

Next, we compute L_n.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{\sum_{i=1}^{n}(i-1)^2}{n^3} = \frac{1}{n^3}\cdot\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}$$

$$= \frac{1}{n^3}\cdot\frac{(n-1)n\cdot(2n-1)}{6}$$

$$= \frac{(n-1)(2n-1)}{6n^2}$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to\infty}\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{1}{6}$$

The above method also motivates a way to compute the distance travelled by a particle with a velocity function $v(t)$:

We cut the time interval $[a,b]$ into n pieces of equal length. When $[a,b]$ is divided into many piecies, each piece is a small period, so $v(t)$ won't vary much on each piece. We then regard each piece of time as a travel of constant velocity, by picking a velocity in it, and use it to estimate the distance travelled in the small piece of time. Finally we add them up to obtain an estimation of the total distance. As n is bigger, the pieces are smaller, so the estimation is better. Taking the limit $n\to\infty$, we get the distance travelled during $[a,b]$ is

$$d = \lim_{n\to\infty} \sum v(t_i)\Delta t.$$