ANTIDERIVATIVES

Definition. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example. If $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an antiderivative of f.

Also note that for any constant C. $F(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f, since $\frac{d}{dx}(\frac{1}{3}x^3 + C) = x^2$.

In general, we can do this for the antiderivative of any function:

Theorem. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is arbitrary constant

Example. Find the most general antiderivative of $f(x) = \sin x$

If $F(x) = -\cos x$, $F'(x) = \sin x$

So the most general antiderivative of f is

$F(x) = -\cos x + C$

Proposition. If $F(x)$ is an antiderivative of $f(x)$, $G(x)$ is an antiderivative of $g(x)$:
(i) $CF(x)$ is an antiderivative of $Cf(x)$
(ii) $F(x) \pm G(x)$ is an antiderivative of $f(x) \pm g(x)$

Example. Find the most general antiderivative of $f(x) = \frac{1-\cos^3 x}{1-\sin^2 x}$

$$f(x) = \frac{1-\cos^3 x}{1-\sin^2 x} = \frac{1-\cos^3 x}{\cos^2 x} = \frac{1}{\cos^2 x} - \cos x$$

$F(x) = \tan x - \sin x + C.$

We've seen that for a given function f, there're many antiderivatives: $F(x)+C$ for each constant C.

But sometimes we are looking for an antiderivative with some extra condition, in this case, the constant C is often determined by the extra condition.

Example. Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

$$f'(x) = e^x + \frac{20}{1+x^2}.$$

so $f(x) = e^x + 20\tan^{-1}x + C$.

$-2 = f(0) = e^0 + 20\cdot\tan^{-1}0 + C \Rightarrow -2 = 1 + 0 + C \Rightarrow C = -3$

so $f(x) = e^x + 20\tan^{-1}x - 3$

Example. Find $f(x)$ if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.

$f''(x) = 12x^2 + 6x - 4$, so $f'(x) = 4x^3 + 3x^2 - 4x + C_1$

$f(x) = x^4 + x^3 - 2x^2 + C_1x + C_2$

$$\begin{cases} 4 = f(0) = C_2 \\ 1 = f(1) = 1 + 1 - 2 + C_1 + C_2 \end{cases} \Rightarrow \begin{cases} C_1 = -3 \\ C_2 = 4 \end{cases}$$

so $f(x) = x^4 + x^3 - 2x^2 - 3x + 4$

Example. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6\,m/s$ and its initial displacement is $s(0) = 9\,m$. Find its position function $s(t)$.

Recall that $v(t) = s'(t)$ and $a(t) = v'(t)$.

$a(t) = 6t+4,\quad$ so $\quad v(t) = 3t^2+4t+C_1.$

$-6 = v(0) = C_1.\quad$ So $\quad v(t) = 3t^2+4t-6$

then $\quad s(t) = t^3+2t^2-6t+C_2$

$9 = s(0) = C_2.\quad$ so $\quad s(t) = t^3+2t^2-6t+9$

Example. A ball is thrown upward with speed 4.9m/s from the edge of a cliff 9.8 m. above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? (From Newtonian physics, the acceleration of the ball is 9.8 m/s^2 pointing downward).

$a(t) = -9.8,\quad$ so $\quad v(t) = -9.8t + C$

$4.9 = v(0) = C_1\quad$ so $\quad v(t) = -9.8t+4.9$

It reaches maximum height when $v(t) = s'(t) = 0$.

Let $\quad v(t) = -9.8t+4.9=0\quad$ we get $\quad t=0.5$

so it reaches maximum height at 0.5 second.

$s(t) = -4.9t^2+4.9t+C_2.$

$9.8 = s(0) = C_2 \Rightarrow s(t) = -4.9t^2+4.9t+9.8$

Let $\quad s(t) = -4.9t^2+4.9t+9.8=0$

$$-4.9(t^2-t-2)=0$$

$$-4.9(t+1)(t-2)=0$$

So $\quad t=2\quad$ or $\quad t=-1$ (drop).

we see it hit the ground at 2 second.