CURVE SKETCHING

The basic steps to sketch the curve of a function $f(x)$:

① Domain.
② Intercepts.
③ Symmetry
④ Asymptotes.
⑤ Intervals of Increase & Decrease
⑥ Local Extreme values.
⑦ Concavity and Points of Inflection
⑧ Sketch the Curve.

Example. Sketch $y = \frac{2x^2}{x^2-1}$

Domain: $x^2 - 1 \neq 0 \Rightarrow x \neq \pm 1$.

Intercept: $f(0) = 0$, so the x-intercept and y-intercept are both 0.

Symmetry: This is an even function.

Asymptotes: $\lim\limits_{x\to+\infty} \frac{2x^2}{x^2-1} = \lim\limits_{x\to-\infty} \frac{2x^2}{x^2-1} = 2$.

so $y=2$ is the horizontal asymptote.

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = +\infty, \quad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty.$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty, \quad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = +\infty$$

Increasing & Decreasing: $y' = \frac{-4x}{(x^2-1)^2}$

so f is increasing on $(-\infty, -1)$ and $(-1, 0)$
decreasing on $(0, 1)$ and $(1, +\infty)$

Local extrema: The only critical number is $x=0$.

We see f' changes from positive to negative at 0, $f(0)=0$ is a local maximum

Concavity & Inflection Points: $f''(x) = \dfrac{12x^2+4}{(x^2-1)^3}$

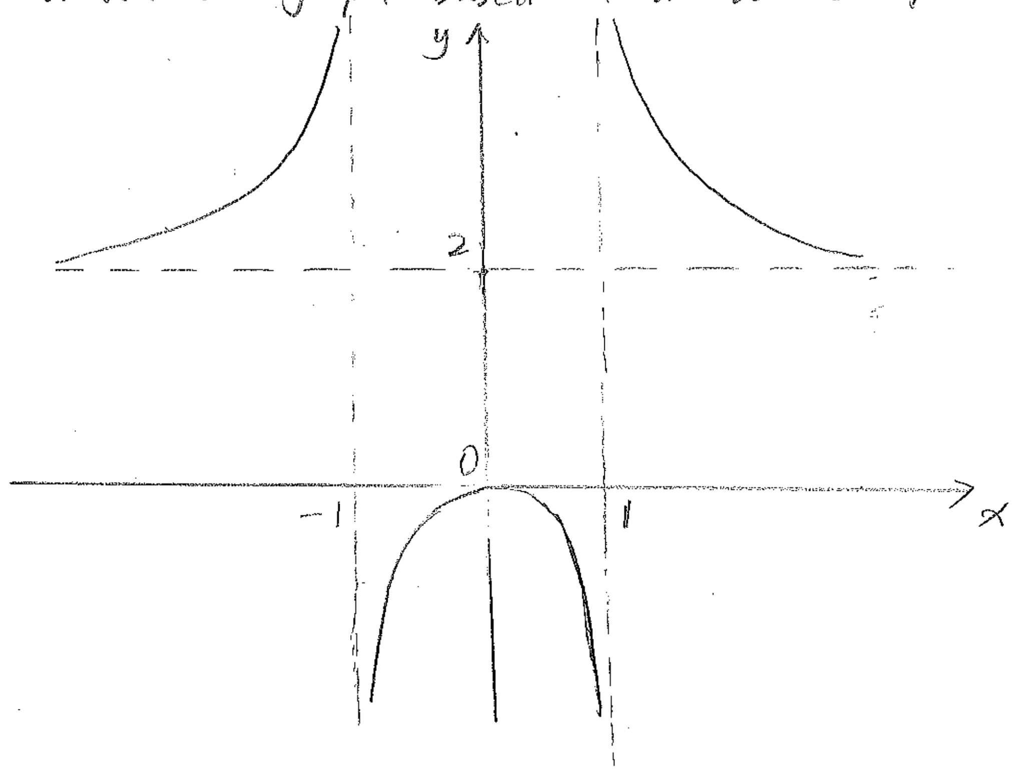
We see $f''(x)>0$ on $(-\infty,-1)$ and $(1,+\infty)$.

$f''(x)<0$ on $(-1,1)$

So f is concave upward on $(-\infty,-1)$, $(1,+\infty)$

concave downward on $(-1,1)$.

We sketch the graph based on the above information.

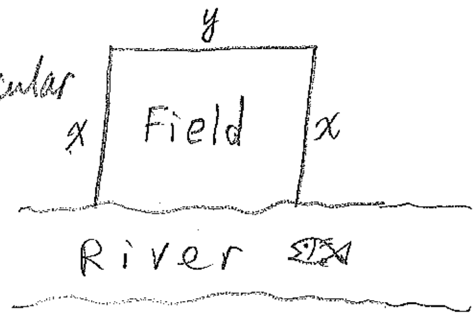


OPTIMIZATION

Theorem (First Derivative Test For Absolute Extreme Values)
Suppose that c is a critical number of a continuous function f defined on an interval.

(a). If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
(b). If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f

Example. A farmer has 2400 m. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



Let the length of the edge perpendicular to river to be x, the length of the edge parallel to x to be y.

Then then area of the field is $A = xy$.

Note $2x + y = 2400$. so $y = 2400 - 2x$

$$A = x(2400 - 2x) = 2400x - 2x^2$$

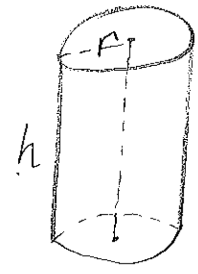
$\frac{dA}{dx} = 2400 - 4x$. The critical number is $x = 600$.

we see $\frac{dA}{dx} > 0$ for $x < 600$ and $\frac{dA}{dx} < 0$ for $x > 600$

so A obtains absolute maximum when $x = 600$.
the corresponding area is $A = 600 \times (2400 - 2 \times 600) = 7.2 \times 10^5$

Example. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Let r be the radius and h be the height for the cylinder.



The Total surface area is

$$S = 2\cdot\pi r^2 + 2\pi r h$$

Observe the volume of the cylinder is

$$1 = \pi r^2 h \implies h = \frac{1}{\pi r^2}$$

So $\quad S = 2\pi\cdot r^2 + 2\pi r\cdot\frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$

$$S' = 4\pi r - \frac{2}{r^2} = \frac{4\pi r^3 - 2}{r^2}$$

let $S' = 0$. we get $\quad r = \frac{1}{\sqrt[3]{2\pi}}$

and $\quad S' < 0 \quad$ when $\quad 0 < r < \frac{1}{\sqrt[3]{2\pi}}$,

$\quad\quad S' > 0 \quad$ when $\quad r > \frac{1}{\sqrt[3]{2\pi}}$

We conclude S is minimized when $r = \frac{1}{\sqrt[3]{2\pi}}$

and $\quad h = \frac{1}{\pi r^2} = \frac{(2\pi)^{\frac{2}{3}}}{\pi} = 2\cdot\frac{2^{-\frac{1}{3}}\cdot\pi^{\frac{2}{3}}}{\pi} = 2\cdot\frac{1}{(2\pi)^{\frac{1}{3}}} = \frac{2}{\sqrt[3]{2\pi}}$

Example. In Economics, people are interested in maximizing profit. A simple model for studying profit is as follows:

Let x be the production of certain commodity.

$C(x)$ is the cost function, i.e the cost of producing x units.

$p(x)$ is the demand function (price function). i.e. the price of the goods if x units will be sold.

$R(x) = xp(x)$ is called the revenue function.

$P(x) = R(x) - C(x)$ is called the profit function.

We call $C'(x)$ the marginal cost, $R'(x)$ the marginal revenue, and $P'(x)$ the marginal profit.

When profit $P(x)$ is maximized, we know $P'(x) = 0$.

But $P'(x) = R'(x) - C'(x)$. $\Rightarrow R'(x) = C'(x)$ when x maximizes the profit. So we conclude profit is maximized when marginal cost equals marginal revenue.

Let's do a more concrete example:

A store has been selling 200 DVD burners a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize revenue?

The price function is $p(x) = 350 - \frac{10}{20}(x - 200) = -\frac{1}{2}x + 450$

The revenue function is $R(x) = x\,p(x) = -\frac{1}{2}x^2 + 450x$

$R'(x) = -x + 450$

we see $R'(450) = 0$. $R'(x) > 0$ when $x < 450$

$R'(x) < 0$ when $x > 450$

So $R(x)$ has maximum $R(450) = 225$.

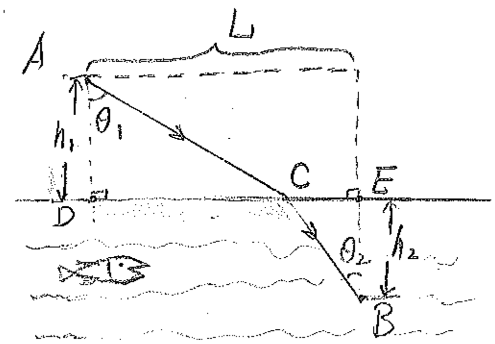
the rebate is $\$350 - \$225 = \$125$

Example. v_1 is the speed of light in air, v_2 is the speed of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.

Let h_1 be the altitude of A above water, h_2 be the altitude of B below water. L is the horizontal distance between A & B.



Then $h_1 \tan\theta_1 + h_2 \tan\theta_2 = L$. --- $(*)$

let T be the time taken for the light from A to B.

$$T = \frac{\frac{h_1}{\cos\theta_1}}{V_1} + \frac{\frac{h_2}{\cos\theta_2}}{V_2} = \frac{h_1}{V_1\cos\theta_1} + \frac{h_2}{V_2\cos\theta_2} \quad \text{---} \ (**)$$

Regard θ_2 as a function of θ_1.

Using Implicit Differentiation to $(*)$.

$$h_1 \cdot \frac{1}{\cos^2\theta_1} + h_2 \cdot \frac{1}{\cos^2\theta_2} \cdot \frac{d\theta_2}{d\theta_1} = 0$$

we get $\displaystyle \frac{d\theta_2}{d\theta_1} = -\frac{h_1\cos^2\theta_2}{h_2\cos^2\theta_1}$

Now differentiate $(**)$

$$\frac{dT}{d\theta_1} = -\frac{h_1}{V_1}\cdot\frac{1}{\cos^2\theta_1}\cdot(-\sin\theta_1) - \frac{h_2}{V_2}\cdot\frac{1}{\cos^2\theta_2}\cdot(-\sin\theta_2)\cdot\frac{d\theta_2}{d\theta_1}$$

$$= \frac{h_1}{V_1}\cdot\frac{\sin\theta_1}{\cos^2\theta_1} - \frac{h_2}{V_2}\cdot\frac{\sin\theta_2}{\cos^2\theta_2}\cdot\frac{h_1\cos^2\theta_2}{h_2\cos^2\theta_1}$$

$$= \frac{h_1}{V_1}\cdot\frac{\sin\theta_1}{\cos^2\theta_1} - \frac{h_1}{V_2}\frac{\sin\theta_2}{\cos^2\theta_1}$$

$$= \frac{h_1}{\cos^2\theta_1}\cdot\left(\frac{\sin\theta_1}{V_1} - \frac{\sin\theta_2}{V_2}\right)$$

Let $\frac{dT}{d\theta_1} = 0$, we get $\frac{\sin\theta_1}{V_1} - \frac{\sin\theta_2}{V_2} = 0$, i.e. $\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

If θ_1 is larger than that, $\frac{dT}{d\theta_1} > 0$.

If θ_1 is smaller than that, $\frac{dT}{d\theta_1} < 0$

We conclude T is minimized when $\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$