DERIVATIVES AND THE SHAPES OF GRAPHS

Theorem. (Increasing / Decreasing Test)
If $f'(x)>0$ on an interval, then f is increasing on it.
If $f'(x)<0$ on an interval, then f is decreasing on it.

Geometrically, the intuition is clear: if $f'(x)>0$, then the slope is positive, which indicates the graph of f is going "upward" as x increases.

Proof. If $f'(x)>0$ on an interval and $x_1<x_2$ are two numbers in this interval, then by the Mean Value Theorem, there exists c in (x_1, x_2) such that

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)>0. \Rightarrow f(x_2)>f(x_1)$$

Similarly we can prove the case for $f'(x)<0$.

Example. Find where the function $f(x)=3x^4-4x^3-12x^2+5$ is increasing and where it's decreasing.

$$f'(x)=12x^3-12x^2-24x=12x(x^2-x-2)$$
$$=12x(x-2)(x+1)$$

Let $f'(x)>0$. we get $-1<x<0$ or $x>2$.
Let $f'(x)<0$. we get $x<-1$ or $0<x<2$.

So f is increasing on $(-1,0)$ and $(2,+\infty)$
decreasing on $(-\infty,-1)$ and $(0,2)$

The Increasing / Decreasing Test can be applied to determine if a critical number gives a local extreme

Theorem (The First Derivative Test)
Suppose that c is a critical number of a continuous function f.

(a). If $f'(x)$ changs from positive to negative at c, then f has a local maximum at c.

(b). If $f'(x)$ changes from negative to positive at c, then f has a local minimum.

(c). If $f'(x)$ doesn't change sign at c, then f has no local maximum or minimum at c.

Example. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

We've computed in the previous example that

$f'(x) = 12x(x-2)(x+1)$, the critical numbers are $-1, 0, 2$.

The sign of $f'(x)$ is :

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	<0	>0	<0	>0

So $f(x)$ obtains local maximum $f(0) = 5$
local minimum $f(-1) = 0$ and $f(2) = -27$

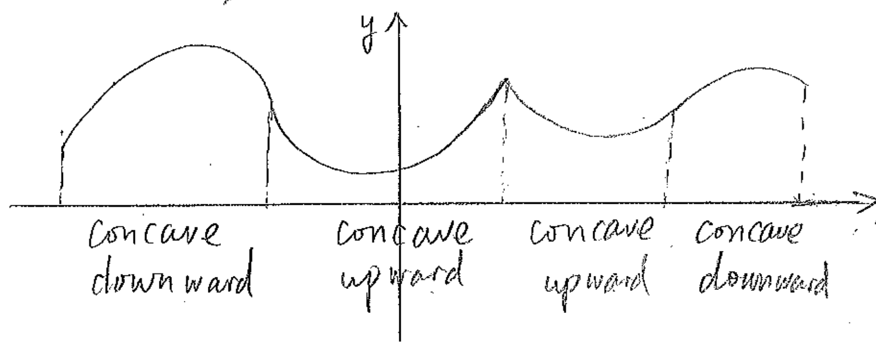
Example. $f(x) = x^3$.

$f'(x) = 3x^2$. The critical number is $x=0$.

But $f'(x)>0$ for both $x<0$ and $x>0$, i.e. f' doesn't change sign at $x=0$. we conclude $f(0)$ is not a local extreme.

Definition. If the graph of f lies above all of its tangents on an interval I, then it's called concave upward on I.

If the graph of f lies below all of its tangents on an interval I, then it's called concave downward on I.



Remark. In some other books, people use the terms "convex" for concave upward and "concave" for concave downward.

Definition. A point P on a curve $y=f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward.

Theorem (Concavity Test)

(a). If $f''(x)>0$ for all x in I, then the graph of f is concave upward on I.

(b). If $f''(x)<0$ for all x in I, then the graph of f is concave downward on I.

Corollary. There is a point of inflection at any point where the second derivative changes sign.

Example. Find the inflection points of $f(x) = x^3 - 12x + 2$ and the intervals on which f is concave upward / downward

$f'(x) = 3x^2 - 12 \qquad f''(x) = 6x$

we see $f''(0) = 0$, $f''(x) < 0$ for $x < 0$,
$f''(x) > 0$ for $x > 0$.

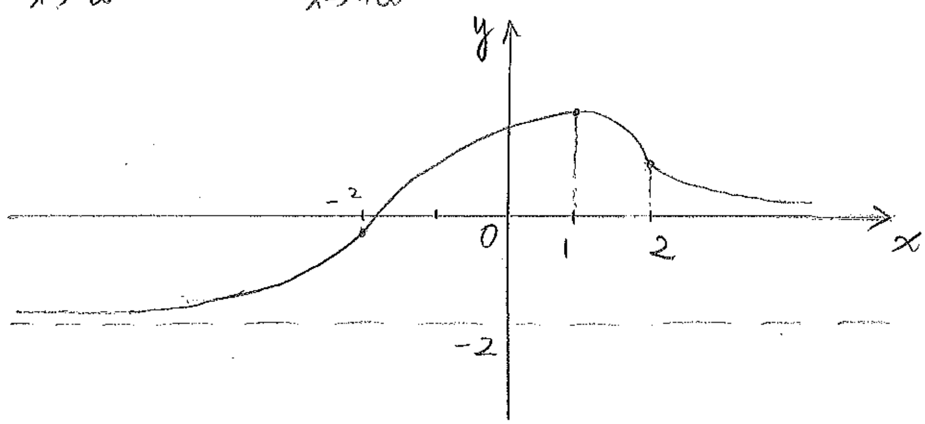
So $x = 0$ is an inflection point,
f is concave upward on $(0, +\infty)$
concave downward on $(-\infty, 0)$

Example. Sketch a possible graph of a function f that satisfies the following conditions:

① $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, +\infty)$

② $f''(x) > 0$ on $(-\infty, -2)$ and $(2, +\infty)$, $f''(x) < 0$ on $(-2, 2)$

③ $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to +\infty} f(x) = 0$



Theorem (The Second Derivative Test)

Suppose f'' is continuous near c.

(a). If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c

(b). If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

Example. Find the local extrema of $f(x)=x^4-4x^3$.

$f'(x)=4x^3-12x^2$. Let $f'(x)=0$, we get the critical numbers are 0 and 3.

$f''(x)=12x^2-24x=12x(x-2)$

So $f''(0)=0$, $f''(3)=36$

$f'(3)=0$ and $f''(3)>0 \Rightarrow f(3)=-27$ is a local minimum

$f'(0)=0$ and $f''(0)=0$. The Second Derivative Test doesn't work. but observe $f'(x)<0$ on both $(-\infty,0)$ and $(0,3)$, so $f'(x)$ doesn't change sign at $x=0$. we conclude $f(0)$ is not a local extreme.