THE MEAN VALUE THEOREM

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following:

① f is continuous on $[a, b]$.

② f is differentiable on (a,b).

③ $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof If f is a constant function, then $f'(x) \equiv 0$ on (a,b).

If f is not a constant function, there exists an absolute extrema c in (a,b), then by Fermat's Theorem, $f'(c) = 0$.

Example Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Let $f(x) = x^3 + x - 1$. First, $f(0) = -1 < 0$, $f(1) = 1 > 0$.
so by Intermediate Value Theorem, there's c in $(0,1)$ such that $f(c) = 0$

Now suppose $f(x) = 0$ at two distinct points $x = a$ and $x = b$.

Then by Rolle's Theorem, there's d between a and b such that $f'(d) = 0$.
But $f'(x) = 3x^2 + 1 \neq 0$ for all x, so $f(x) = 0$ cannot have more than 1 roots.

Theorem (The Mean Value Theorem)

Let f be a function that satisfying:

① f is continuous on $[a, b]$.
② f is differentiable on (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Equivalently, $f(b) - f(a) = f'(c)(b-a)$

Proof. Let $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

Note that $h(a) = 0$, $h(b) = 0$.
and $h(x)$ is continuous on $[a,b]$ and differentiable on (a, b). so by Rolle's Theorem. there's c in (a,b) such that $h'(c)=0$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

so $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. $\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Example. Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can $f(2)$ possibly be?

By Mean Value Theorem, $f(2) - f(0) = f'(c)(2-0)$ for some c in $(0, 2)$.

$f(2) = f(0) + f'(c) \cdot 2 = -3 + 2f'(c) \le -3 + 2 \times 5 = 7$

So the largest possible value for $f(2)$ is 7.

Theorem. If $f'(x)=0$ for all x on (a,b), then f is constant on (a,b)

Proof. Let $x_1<x_2$ be two numbers on (a,b).

By Mean Value Theorem, there's c in (x_1, x_2) such that

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)=0$$

$$\Rightarrow f(x_1)=f(x_2)$$

Since x_1, x_2 are arbitrary, we see f is a constant function on (a,b)

Corollary. If $f'(x)=g'(x)$ for all x in an interval (a,b), then $f-g$ is constant on (a,b).

Proof. Let $h(x)=f(x)-g(x)$, then $h'(x)=0$ for all x in (a,b). By the previous Theorem, $h(x)$ is a constant function on (a,b)