MAXIMUM AND MINIMUM VALUES

Definition. Let c be a number in the domain D of a function f. Then $f(c)$ is the:

- Absolute maximum value of f on D if $f(c) \geq f(x)$ for all $x \in D$
- Absolute minimum value of f on D if $f(c) \leq f(x)$ for all $x \in D$

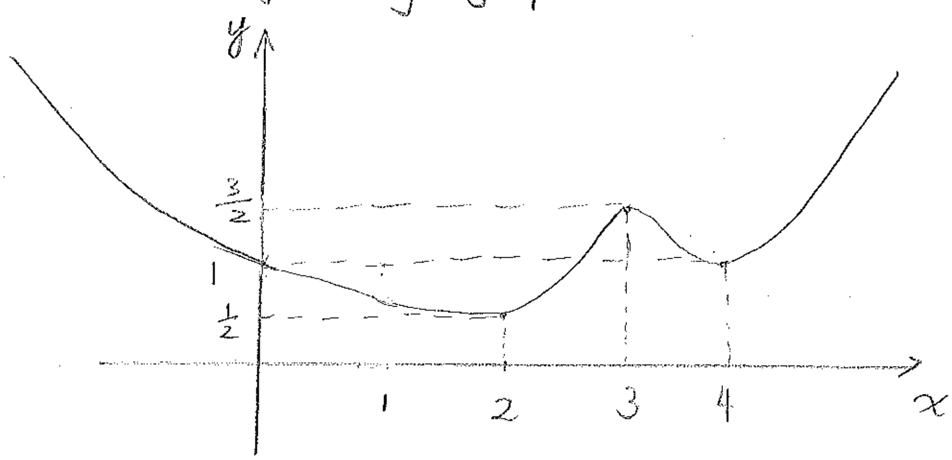
Sometimes, we also call them global maximum/minimum. The maximum and minimum values are called extreme values of f.

Definition. The number $f(c)$ is a

- Local maximum value of f if $f(c) \geq f(x)$ when x is near c
- Local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Example. $f(x) = \cos x$ takes local maximum value 1 at $x = 2k\pi$, $k \in \mathbb{Z}$ and 1 is also the absolute maximum value for $f(x) = \cos x$

Example. $f(x)$ has the following graph:



The $f(x)$ has local maximum value $\frac{3}{2}$ at $x = 3$.
local minimum value $\frac{1}{2}$ at $x = 2$.
local minimum value 1 at $x = 4$
$f(x)$ has absolute mimimum value $\frac{1}{2}$ at $x = 2$.
and $f(x)$ doesn't have absolute maximum value.

We see in the previous example that in general absolute maximum/minimum may not exist.

But if we add some assumption, we can always obtain them:

Theorem (The Extreme Value Theorem) If f is a continuous function on a closed interval $[a,b]$, then f attains absolute maximum value $f(c)$ and absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

The above theorem only tells us about the existence, but not a method to explicitly find the absolute extrema. We're going to see how to find them.

Theorem (Fermat's Theorem) If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.

Proof. If f has a local maximum at c, then $f(x) \le f(c)$ for all x near c.

So $\lim\limits_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le 0$, $\lim\limits_{x\to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$.

If $f'(c)$ exists, then $f'(c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$,

so $f'(c) \le 0$ and $f'(c) \ge 0$, we get $f'(c)=0$

Remark. The converse of the Fermat's Theorem is not correct in general.

For example, $f(x)=x^3$, $f'(x)=3x^2$. we see $f'(0)=0$, but $f(0)=0$ is not a local extreme.

Definition. A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ doesn't exist.

Example. Find the critical numbers of $f(x)=x^{\frac{3}{5}}(4-x)$

$$f'(x) = x^{\frac{3}{5}}\cdot(-1)+\frac{3}{5}x^{-\frac{2}{5}}(4-x) = -x^{\frac{3}{5}}+\frac{12}{5}x^{-\frac{2}{5}}-\frac{3}{5}x^{\frac{3}{5}}$$
$$= -\frac{8}{5}x^{\frac{3}{5}}+\frac{12}{5}x^{-\frac{2}{5}}$$
$$= \frac{-8x+12}{5x^{\frac{2}{5}}}$$

$f'(x)=0$ when $x=\frac{3}{2}$.

$f'(x)$ doesn't exist when $x=0$

So the critical numbers are $x=0$ and $x=\frac{3}{2}$

Proposition. If f has a local extreme at c, then c is a critical number.

Now we have a method to find the absolute extrema:

The Closed Interval Method: To find the absolute extrema of a continuous function on $[a,b]$:

① Find the values of f at cricitical numbers in $[a,b]$

② Find $f(a)$ and $f(b)$

③ The largest among those in ①, ② is the absolute maximum.
The smallest among those in ①, ② is the absolute minimum

Example. $f(x)=x^3-3x^2+1$. Find the absolute maximum and minimum of $f(x)$ on $[-\frac{1}{2}, 4]$.

$f'(x)=3x^2-6x$ let $f'(x)=0$ we get $x=0$ or $x=2$.

both of them are in the interval $[-\frac{1}{2},4]$, they're critical numbers.

$f(0) = 1$, $f(2) = -3$.

Next consider the two endpoints:

$f(-\frac{1}{2}) = \frac{1}{8}$, $f(4) = 17$.

So compare the above values, we see

$f(x)$ on $[-\frac{1}{2}, 4]$ has absolute maximum value $f(4) = 17$

and absolute minimum value $f(2) = -3$