Inverse Trigonometric Functions

Trigonometric functions are in general not one-to-one, but we may set some convention to restrict our attention on smaller intervals on which they're one-to-one, then we can talk about the inverse trigonometric functions

- $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is one-to-one with image $[-1, 1]$. so it has an inverse function $f^{-1}(x) = \sin^{-1}(x)$ defined on $[-1, 1]$ with range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

  by the definition of inverse function, we see

  $$\sin^{-1} x = y \iff \sin y = x, \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

  In other words, $\sin^{-1} x$ stands for the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine value is x.

Example. $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$, since $\sin\frac{\pi}{6} = \frac{1}{2}$ and $-\frac{\pi}{2} \le \frac{\pi}{6} \le \frac{\pi}{2}$

Proposition (Cancellation).
- $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
- $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$

Example. $\sin^{-1}(\sin\frac{\pi}{5}) = \frac{\pi}{5}$. $\sin(\sin^{-1}\frac{1}{10}) = \frac{1}{10}$.

Example.
- $\sin^{-1}(\sin\pi) = \sin^{-1} 0 = 0$.
- $\sin^{-1}(\sin\frac{7\pi}{6}) = \sin^{-1}(\sin(-\frac{\pi}{6})) = -\frac{\pi}{6}$.

Note the last example is a tricky one. When x is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the cancellation law doesn't hold. We need to find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is the same as the given angle to be $\sin^{-1}(\sin x)$

We can similarly define the inverse cosine function $\cos^{-1}x$ and inverse tangent function $\tan^{-1}x$:

- $f(x) = \cos x$ on $[0, \pi]$ is one-to-one with image $[-1, 1]$, so it has an inverse function $f^{-1}(x) = \cos^{-1}x$ defined on $[-1, 1]$ with image $[0, \pi]$.
- $f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ is one-to-one with image $(-\infty, +\infty)$, so it has an inverse function $f^{-1}(x) = \tan^{-1}x$ defined on $(-\infty, +\infty)$ with image $(-\frac{\pi}{2}, \frac{\pi}{2})$.

So $\cos^{-1}x = y \iff x = \cos y$ and $0 \le y \le \pi$

$\tan^{-1}x = y \iff x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example. $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $0 \le \frac{\pi}{4} \le \pi$.

Example. Simplify $\cos(\tan^{-1}x)$

let $y = \tan^{-1}x$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
then $x = \tan y$.

this implies $\sec^2 y = 1 + \tan^2 y = 1 + x^2$.

So $\cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$.

since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we know $\cos y > 0$.

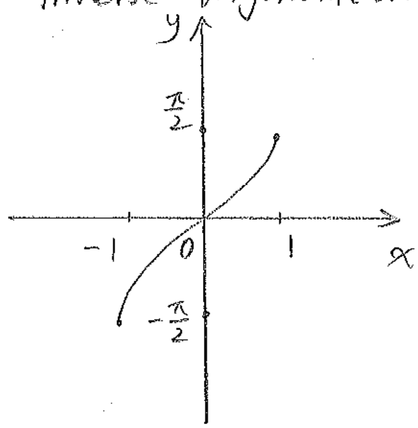
we see $\cos y = \frac{1}{\sqrt{1+x^2}}$

i.e. $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

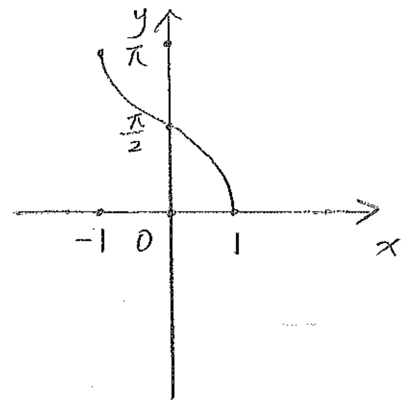
Proposition (Cancellation)

- $\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$
- $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$
- $\tan(\tan^{-1}x) = x$ for any x.
- $\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

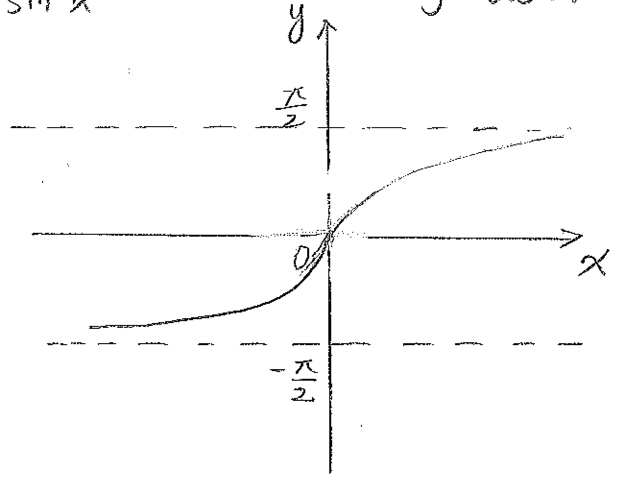
Graphs of inverse trigonometric functions:



$y = \sin^{-1}x$



$y = \cos^{-1}x$



$y = \tan^{-1}x$

Note that for the inverse tangent function,

$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x \to +\infty} \tan^{-1}x = \frac{\pi}{2}.$$

The lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes.

Now we study the derivatives of inverse trigonometric functions.

Theorem. $(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$, $(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$, $(\tan^{-1}x)' = \frac{1}{1+x^2}$

Proof. They can be shown by using the formula $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

We'll show the first one here and leave the other two as exercises.

Let $f(x) = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $f^{-1}(x) = \sin^{-1}x$

$$(\sin^{-1}x)' = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1}x)}$$

let $y = \sin^{-1}x$, we get $x = \sin y$.

$$\cos^2 y = 1 - \sin^2 y = 1 - x^2.$$

Since $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $\cos y \ge 0$, so $\cos y = \frac{1}{\sqrt{1-x^2}}$

We conclude $(\sin^{-1}x)' = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

Example. Differentiate $f(x) = x\tan^{-1}\sqrt{x}$

$$f'(x) = \tan^{-1}\sqrt{x} + x \cdot \frac{1}{1+(\sqrt{x})^2} \cdot (\sqrt{x})'$$
$$= \tan^{-1}\sqrt{x} + \frac{x}{1+x} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$$
$$= \tan^{-1}\sqrt{x} + \frac{x}{2(1+x)}$$

Remark. Another way to say the inverse trigonometric functions is:

$\arcsin x = \sin^{-1}x$, arcsine function

$\arccos x = \cos^{-1}x$, arccosine function.

$\arctan x = \tan^{-1}x$, arctangent function.

L'Hospital's Rule.

Definition. If we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \longrightarrow \infty$ (or $-\infty$) and $g(x) \longrightarrow \infty$ (or $-\infty$), then the limit may or may not exist and is called an indeterminate form of type $\frac{\infty}{\infty}$.

If both $f(x) \longrightarrow 0$ and $g(x) \longrightarrow 0$, then it's called an indeterminate form of type $\frac{0}{0}$.

One way to compute such limit is the L'Hospital's Rule:

Theorem (L'Hospital's Rule)
Suppose f & g are differentiable. $g'(x) \neq 0$ near a. (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit on the right side exists or $\pm\infty$.

Example. Find $\lim_{x \to 1} \frac{\ln x}{x-1}$.

$\lim_{x \to 1} \ln x = \ln 1 = 0$. $\lim_{x \to 1} (x-1) = 0$.

Using L'Hospital's Rule: $\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{1}{x} = 1$.

Example. Find $\lim_{x \to \infty} \frac{e^x}{x^2}$

$\lim_{x \to \infty} e^x = +\infty$. $\lim_{x \to \infty} x^2 = +\infty$, Using L'Hospital's Rule:

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{(e^x)'}{(x^2)'} = \lim_{x \to \infty} \frac{e^x}{2x}$$

$\lim_{x\to\infty} e^x = +\infty$ and $\lim_{x\to\infty} 2x = +\infty$. Using L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to\infty} \frac{e^x}{2} = +\infty$$

Remark. ① We must check the conditions before applying the L'Hospital's Rule.

For example. $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$ cannot be computed by the L'Hospital's Rule: $\lim_{x\to\pi^-} \sin x = 0$, but $\lim_{x\to\pi^-}(1-\cos x) = 2$

$$\text{so } \lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{\lim_{x\to\pi^-} \sin x}{\lim_{x\to\pi^-} 1-\cos x} = \frac{0}{2} = 0$$

If we forgot to check the conditions and accidentally use the L'Hospital's Rule:

$$\lim_{x\to\pi^-} \frac{(\sin x)'}{(1-\cos x)'} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = \lim_{x\to\pi^-} \frac{1}{\tan x} = -\infty.$$

We'll get the wrong solution.

② For the limit $\lim_{x\to 0} \frac{\sin x}{x}$, it seems we can use the L'Hospital's Rule: $\lim_{x\to 0} \sin x = 0$, $\lim_{x\to 0} x = 0$. so

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \cos x = 1$$

But this logically incorrect! The reason is that if we want to compute $\lim_{x\to 0} \frac{\sin x}{x}$ by L'Hospital's Rule, we make use of the fact $(\sin x)' = \cos x$.

But recall that when we compute the formula $(\sin x)' = \cos x$, we made use of $\lim_{x\to 1} \frac{\sin x}{x} = 1$. So the L'Hospital's Rule brings a cyclic argument on this case.

There are more general applications of the L'Hospital's Rule.

$\lim_{x\to a} f(x)g(x)$ is called an indeterminate form of type $0\cdot\infty$ if

$\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\pm\infty$.

We can compute $\lim_{x\to a} f(x)\,g(x)$ by L'Hospital's Rule to:

$$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}} \qquad \left(\text{Note } \lim_{x\to a} f(x)=\lim_{x\to a}\frac{1}{g(x)}=0\right)$$

Example. Compute $\lim_{x\to 0^+} x\ln x$

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{(\ln x)'}{(\frac{1}{x})'}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

$\lim_{x\to a}[f(x)-g(x)]$ is called an indeterminate form of type $\infty-\infty$

if $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=\infty$.

The strategy is to try to convert $\lim_{x\to a}[f(x)-g(x)]$ into quotient.

Example. $\lim_{x\to(\frac{\pi}{2})^-}(\sec x-\tan x)$.

We see $\lim_{x\to(\frac{\pi}{2})^-}\sec x=\lim_{x\to(\frac{\pi}{2})^-}\frac{1}{\cos x}=+\infty$, $\lim_{x\to(\frac{\pi}{2})^-}\tan x=+\infty$

so we cannot directly tell the limit

$$\lim_{x\to(\frac{\pi}{2})^-}(\sec x-\tan x)=\lim_{x\to(\frac{\pi}{2})^-}\frac{1}{\cos x}-\frac{\sin x}{\cos x}=\lim_{x\to(\frac{\pi}{2})^-}\frac{1-\sin x}{\cos x}$$

$$=\lim_{x\to(\frac{\pi}{2})^-}\frac{-\cos x}{-\sin x}$$

$$=0$$

$\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form in the following cases.

- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$: type 0^0
- $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$: type ∞^0
- $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$: type 1^∞

We can use the logarithmic differentiation to translate it into an indeterminate form of $0\cdot\infty$, and then apply the L'Hospital's Rule.

Example. Find $\lim_{x\to 0^+} x^x$: $\lim_{x\to 0^+} x = 0$, so it's of form 0^0

$$\ln x^x = x \ln x$$

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

so $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = e^0 = 1$

Example. Find $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

$\lim_{x\to 0^+} (1+\sin 4x) = 1$, $\lim_{x\to 0^+} \cot x = +\infty$

so it's of form 1^∞.

$$\ln (1+\sin 4x)^{\cot x} = \cot x \cdot \ln(1+\sin 4x)$$

$\lim_{x\to 0^+} \cot x = +\infty$. $\lim_{x\to 0^+} \ln(1+\sin 4x) = 0$

$$\lim_{x\to 0^-} \cot x \cdot \ln(1+\sin 4x) = \lim_{x\to 0^-} \frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^-} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = \lim_{x\to 0^-} \frac{4\cos 4x \cos^2 x}{1+\sin 4x} = 4$$

so $\lim_{x\to 0^-} (1+\sin 4x)^{\cot x} = \lim_{x\to 0^-} e^{\ln(1+\sin 4x)^{\cot x}} = e^4$