THE FUNDAMENTAL THEOREM OF CALCULUS.

Theorem (The Fundamental Theorem of Calculus)

Suppose $f(x)$ is a continuous function on $[a, b]$.

(1). $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

(2). $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f.

We have proved Part (2) in the previous section. We're going to discuss about Part (1).

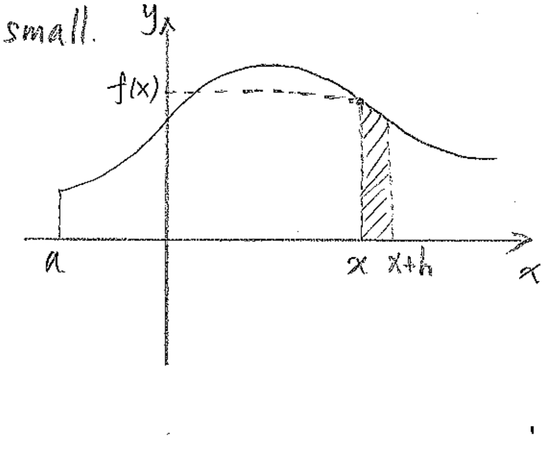
Let $g(x) = \int_a^x f(t)\,dt$, Part (1) tells us that $\frac{d}{da} g(x) = f(x)$.

i.e. $g(x)$ is an antiderivative of $f(x)$.

First we see a geometric explanation:

$\int_a^x f(t)\,dt$ is the signed area bounded by $f(t)$ and between $[a, x]$.

If we increase x to $x+h$, the increase in the area is $\int_x^{x+h} f(t)\,dt \approx h \cdot f(x)$ when h is small.



Then $\frac{\int_x^{x+h} f(t)\,dt}{h} \approx f(x)$.

i.e. $f(x) \approx \frac{g(x+h) - g(x)}{h}$ when h is small, which indicates

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

Now we provide an algebraic proof of Part (1):

$$\frac{g(x+h)-g(x)}{h} = \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h} = \frac{\int_x^{x+h} f(t)\,dt}{h}$$

We may assume $h>0$. The other case can be argued similarly.

Consider the continuous function f on the closed interval $[x, x+h]$

We know it has absolute maximum M and absolute minimum m.

i.e. $m \leq f(t) \leq M$ for t in $[x, x+h]$.

then $\quad mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$.

assume f attains absolute maximum at u in $[x, x+h]$

absolute minimum at v in $[x, x+h]$.

we get $f(v)h \leq \int_x^{x+h} f(t)\,dt \leq f(u)h$

$$f(v) \leq \frac{\int_x^{x+h} f(t)\,dt}{h} \leq f(u)$$

$$f(v) \leq \frac{g(x+h)-g(x)}{h} \leq f(u) \quad \text{----} \ (*)$$

If we take $h \to 0$, since $x \leq u \leq x+h$, $x \leq v \leq x+h$,

we see $u \to x$, $v \to x$, so $f(u) \to f(x)$, $f(v) \to f(x)$

$(*)$ implies $\lim\limits_{h\to 0} f(v) \leq \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h} \leq \lim\limits_{h\to 0} f(u)$

$$f(x) \leq g'(x) \leq f(x)$$

$\Rightarrow \quad g'(x) = f(x)$.

Example. Find $\frac{d}{dx}\int_1^{x^4} \sec t\, dt$.

Let $g(x) = \int_1^x \sec t\, dt$, then $\int_1^{x^4} \sec t\, dt = g(x^4)$.

$$\frac{d}{dx} g(x^4) = \frac{d\,g(u)}{du} \cdot \frac{du}{dx} \quad \text{where } u = x^4. \quad \text{by the Chain Rule.}$$

$$= (\sec u)\, 4x^3$$

$$= (\sec x^4)\, 4x^3$$

Definition. We define the average value of f on the interval $[a,b]$ to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example. Find the average value of $f(x) = 1 + x^2$ on $[-1, 2]$.

$$f_{ave} = \frac{1}{2-(-1)}\int_{-1}^{2} (1+x^2)\,dx = 2$$

Remark. It'll be clear why we define the f_{ave} in this manner if you consider the average velocity for a given velocity function.

Theorem. (The Mean Value Theorem For Integrals)
If f is continuous on $[a,b]$, then there exists c in $[a,b]$ such that $f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

i.e. $\int_a^b f(x)\,dx = f(c)(b-a)$

Proof. Let $F(x) = \int_a^x f(t)\,dt$.
The Mean Value Theorem tells us there's c in $[a,b]$ such that

$$F(b) - F(a) = F'(c)(b-a)$$

$$\int_a^b f(t)\,dt = f(c)(b-a)$$