DEFINITE INTEGRAL AND ANTIDERIVATIVE

Theorem. If f is continuous on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

where F is an antiderivative of f (i.e. $F' = f$)

Proof. We divide $[a,b]$ into n subintervals, with endpoints $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. The length of the interval $[x_{i-1}, x_i]$ is $\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$

On the interval $[x_{i-1}, x_i]$, we apply the Mean Value Theorem to the function F:

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(x_i^*)(x_i - x_{i-1}) \\ &= f(x_i^*)(x_i - x_{i-1}) \end{aligned}$$

for some x_i^* in (x_{i-1}, x_i).

$$\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

$$\Rightarrow \quad F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}$$

Taking the limit above as $n \to +\infty$.

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = \int_a^b f(x)\,dx.$$

Example. Evaluate $\int_1^3 e^x\,dx$.

An antiderivative of $f(x) = e^x$ is $F(x) = e^x$.

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e^1 = e^3 - e$$

Example Find the area under the cosine curve from 0 to b, where $0 \le b \le \frac{\pi}{2}$.

An antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$

so $A = \int_0^b \cos x \, dx = \sin x \Big]_0^b = \sin b - \sin 0 = \sin b$

Definition. Define the indefinite integral of f to be $\int f(x)\,dx = F(x)$, where $F'(x) = f(x)$, i.e. $F(x)$ is an antiderivative of $f(x)$

The relation between indefinite integral and definite integral of f is indicated by the previous theorem:

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

Example. Evaluate $\int_0^3 (x^3 - 6x)\,dx$

$\int x^3 - 6x \, dx = \frac{1}{4}x^4 - 3x^2$, so

$$\int_0^3 (x^3 - 6x)\,dx = \frac{1}{4}x^4 - 3x^2 \Big]_0^3 = \left(\frac{3^4}{4} - 3 \times 3^2\right) - 0 = -\frac{27}{4}$$

Example. Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

$$\int \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int 2 + t^{\frac{1}{2}} - \frac{1}{t^2}\,dt$$
$$= 2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}.$$

so $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = 2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}\Big]_1^9 = \frac{292}{9}$

Another way to view the theorem we introduced today is to replace $f(x)$ by $F'(x)$:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Theorem (Net Change Theorem)

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Example. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$.

(a). Find the displacement of the particle during $1 \leq t \leq 4$.

(b). Find the distance travelled during $1 \leq t \leq 4$.

(a). The displacement is

$$\int_1^4 v(t)\,dt = \int_1^4 t^2 - t - 6 = \tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 - 6t\Big]_1^4 = -\tfrac{9}{2}$$

(b). The distance travelled is

$$\int_1^4 |v(t)|\,dt = \int_1^4 |t^2 - t - 6|\,dt$$

We know $t^2 - t - 6 = (t-3)(t+2)$.

So $v(t) < 0$ when $-2 < t < 3$, $v(t) > 0$ when $t < -2$ or $t > 3$

$$\begin{aligned}\int_1^4 |t^2 - t - 6|\,dt &= \int_1^3 |t^2 - t - 6|\,dt + \int_3^4 |t^2 - t - 6|\,dt \\ &= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt \\ &= \left(-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right)\Big]_1^3 + \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big]_1^3 \\ &= \frac{61}{6}\end{aligned}$$

We use $|v(t)|$ in the integrand because $|v(t)|$ is the rate of change of the distance.