THE DEFINITE INTEGRAL

Recall that in the previous section, we developed a method to compute the area under a curve $y=f(x)$. In particular, we used $\lim\limits_{n\to\infty} R_n$ and $\lim\limits_{n\to\infty} L_n$.

The method can be further generalized.

First, after dividing $[a,b]$ into n subintervals of length $\frac{b-a}{n}$, instead of choosing the right or left endpoint of each $[x_{i-1}, x_i]$, we can actually choose any point x_i^* in $[x_{i-1}, x_i]$:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

A special choice is to take x_i^* in $[x_{i-1}, x_i]$ such that $f(x_i^*)$ is the maximum on $[x_{i-1}, x_i]$. We call it the upper sum.
Similarly, if we choose x_i^* in $[x_{i-1}, x_i]$ to be the one that $f(x_i^*)$ is the minimum on $[x_{i-1}, x_i]$. We call it the lower sum.

Now we are going to generalize it further.

f is a function defined on $[a,b]$. (Now we do not assume f is continuous or positive)

We divide $[a,b]$ into n smaller subintervals by choosing the partition points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

(Now we do not assume $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ to have same length)

We say $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ form a partition of $[a, b]$.

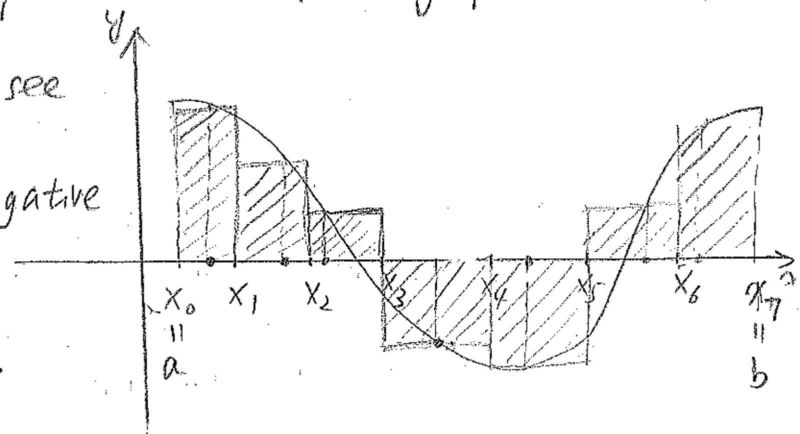
For each $[x_{i-1}, x_i]$, let $\Delta x_i = x_i - x_{i-1}$, which is the length of this interval.

We choose sample points x_i^* from $[x_{i-1}, x_i]$, and define a Riemann Sum associated with f and a partition of $[a, b]$ to be

$$\sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

Geometrically, it corresponds to the following picture:

when $f(x_i^*) < 0$, we see $f(x_i^*) \Delta x_i$ is the negative of the corresponding area of the rectangle.



Observe that although we didn't cut the intervals in an evenly manner, as long as all the subintervals are very small, the region covered by these rectangles is a good approximation of the region bounded by $y = f(x)$ on $[a, b]$. If we define the area above x-axis to be positive and below x-axis to be negative, we see $\sum_{i=1}^{n} f(x_i^*) \Delta x_i$ is a good approximation of the area of the region bounded by $y = f(x)$ on $[a, b]$.

This motivates us to make the following definition:

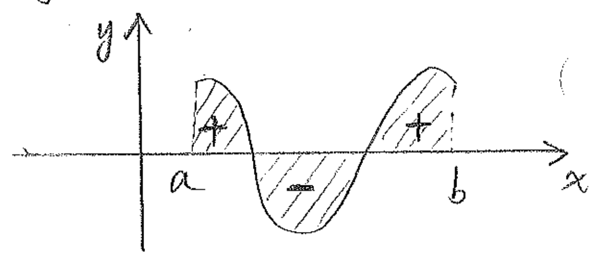
Definition. If f is a function defined on $[a,b]$, the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided that this limit exists. If it does exist, we say f is integrable on $[a,b]$.

We call f the integrand and a, b the limits of integration, and the procedure of calculating an integral is called integration.

Geometrically, the definite integral describes the area of the region between $y=f(x)$ and x-axis, with the conversion that area is positive if $y=f(x)$ is above x-axis and negative if $y=f(x)$ is below x-axis.



Note that $\int_a^b f(x)dx$ may not exist for some $f(x)$. But in some cases, we are sure that a given function is integrable:

Theorem. If f is continuous on $[a,b]$, or f only has finitely many jumping discontinuity points, then f is integrable on $[a,b]$.

Corollary If f and g are functions defined on $[a,b]$, with f continuous on $[a,b]$, $g(x)=f(x)$ except for a finitely many numbers on $[a,b]$, then g is integrable on $[a,b]$.

Computation of Integral in general is a hard problem, we will learn some basic techniques in this course and more powerful and advanced tools will be introduced in Calculus II.

Recall that in the definition of Riemann Sum, there is no restriction on the subdivision of $[a,b]$ and the choice of x_i^* as long as when we take $\max \Delta x_i \to 0$, the Riemann Sum will be the definite integral $\int_a^b f(x)\,dx$. So we may take a good choice of Riemann Sum that is convenient for computation.

Example. Evaluate $\int_0^3 (x^3 - 6x)\,dx$

We may divide $[0,3]$ into n subintervals of size $\frac{3}{n}$ with endpoints $x_0 = 0$, $x_1 = \frac{3}{n}$, $x_2 = \frac{6}{n}$, $\cdots$ $x_i = \frac{3i}{n}$, $\dots$, $x_n = 3$

we see as $n \to \infty$, all $\Delta x_i \to 0$. so $\max \Delta x_i \to 0$.

We take $x_i^* = x_i = \frac{3i}{n}$ then

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{n=1}^{n} f(x_i^*) \Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6 \cdot \frac{3i}{n} \right] \cdot \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\sum_{i=1}^{n} \frac{27 i^3}{n^3} - \sum_{i=1}^{n} \frac{18 i}{n} \right)$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 - \lim_{n\to\infty} \frac{54}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 - \lim_{n\to\infty} \frac{54}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{81}{2} \cdot \left(\frac{n+1}{n}\right)^2 - \lim_{n\to\infty} 27 \cdot \frac{n+1}{n}$$

$$= -\frac{27}{4}$$

Instead of using $x_i^* = x_i$, we can also use $x_i^* = x_{i-1}$ or $x_i^* = \frac{x_i + x_{i-1}}{2}$.

When we try to use a finite Riemann Sum to estimate the actual integral, in many cases $x_i^* = \frac{x_i + x_{i+1}}{2}$ is a good choice.

Theorem (Midpoint Rule)

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{x_i + x_{i-1}}{2}$

Example. Use the Midpoint Rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$

$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$; The 5 subintervals are

$[1, 1.2]$, $[1.2, 1.4]$, $[1.4, 1.6]$, $[1.6, 1.8]$, $[1.8, 2]$.

The midpoints are $1.1, 1.3, 1.5, 1.7, 1.9$

$$\int_1^2 \frac{1}{x}\,dx \approx \left(f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\right)\cdot 0.2$$

$$= \left(\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right)\times 0.2$$

$$\approx 0.691908$$

Definition. If $a > b$, we define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Also we define $\int_a^a f(x)\,dx = 0$

Proposition. Suppose the following integral exists. then:

① $\int_a^b c\,dx = c\cdot(b-a)$, where c is a constant.

② $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

③ $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, where c is a constant

④ If $f(x) \geq g(x)$ for all x on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Corollary. If $f(x) \geq 0$ for all x on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

If $m \leq f(x) \leq M$ for all x on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Example. Consider $\int_0^1 e^{-x^2}\,dx$.

on $[-1, 1]$, $e^{-1} \leq e^{-x^2} \leq e^0 = 1$. so

$$e^{-1}\cdot(1-0) \leq \int_0^1 e^{-x^2}\,dx \leq e^0\cdot(1-0)$$

$$\frac{1}{e} \leq \int_0^1 e^{-x^2}\,dx \leq 1$$