1. \( G \) is a finite group of order \( n \). If \( k \) is an integer that is relatively prime to \( n \), prove the function \( \phi : G \rightarrow G \) defined by \( \phi(x) = x^k \) is a bijective function.

**Solution:** Since \( G \) is finite and \( \phi \) is a function maps \( G \) to itself, it suffices to show \( G \) is surjective.

\( k \) and \( n \) are relatively prime implies there exists \( a, b \in \mathbb{Z} \) such that \( ka + nb = 1 \). For any \( x \in G \), the order of \( x \) divides \( |G| = n \), so \( x^n = 1 \). Thus \( x = x^1 = x^{ka+nb} = (x^a)^k(x^n)^b = (x^a)^k = \phi(x^a) \). We see \( \phi \) is surjective.

2. Let \( a \) and \( b \) be elements of a group \( G \). Prove that \( ab \) and \( ba \) have the same order.

**Solution:** Assume \( |ab| = n \). \( (ab)^n = a(ba)^{n-1}b = 1 \Rightarrow b[a(ba)^{n-1}]a = ba \Rightarrow ba(ba)^{n-1}ba = ba \Rightarrow (ba)^n = 1 \), so \( |ba| \) divides \( |ab| \). Similarly, we can show \( |ba| \) divides \( |ab| \), hence \( |ab| = |ba| \).

3. \( G \) is a group, \( H_1 \) and \( H_2 \) are finite subgroups of \( G \). If \( |H_1| \) and \( |H_2| \) are relatively prime, prove \( H_1 \cap H_2 = \{1\} \).

**Solution:** If \( x \in H_1 \cap H_2 \), then \( x \in H_1 \) implies \( |x| \) divides \( |H_1| \), and \( x \in H_2 \) implies \( |x| \) divides \( |H_2| \). \( |H_1| \) and \( |H_2| \) are relatively prime, so \( |x| = 1 \), i.e. \( x = 1 \).

4. How many different equivalence relations can we define on a set of four elements?

**Solution:** We know equivalence relations are in one-to-one correspondence with partition of a set, so we only need to find all the partitions of a set of four elements.

Denote this set by \( \{a, b, c, d\} \), we see the possible partitions are as follows:

\( \{a\} \sqcup \{b\} \sqcup \{c\} \sqcup \{d\} \),

\( \{a\} \sqcup \{b\} \sqcup \{c, d\} \), \( \{a\} \sqcup \{c\} \sqcup \{b, d\} \), \( \{a\} \sqcup \{d\} \sqcup \{b, c\} \), \( \{b\} \sqcup \{c\} \sqcup \{a, d\} \),

\( \{a, b\} \sqcup \{c, d\} \), \( \{a, c\} \sqcup \{b, d\} \), \( \{a, d\} \sqcup \{b, c\} \)

\( \{a\} \sqcup \{b, c, d\} \), \( \{b\} \sqcup \{a, c, d\} \), \( \{c\} \sqcup \{a, b, d\} \), \( \{d\} \sqcup \{a, b, c\} \)

\( \{a, b, c, d\} \)

So there are in total 15 of them.

Remark: In general, the number of partitions of a set of \( n \) elements is called the Bell Number. You may read this Wikipedia Page for more story on that: https://en.wikipedia.org/wiki/Bell_number

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5. Let \( X = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \} \). Define a relation on \( X \): \( f_1 \sim f_2 \) if \( f_1 - f_2 \equiv C \) for some constant \( C \). Prove this is an equivalence relation on \( X \).

**Solution:** (i). Reflexive: for any \( f \in X \), \( f - f \equiv 0 \), so \( f \sim f \)

(ii). Symmetric: If \( f_1 \sim f_2 \), then there exists \( C \in \mathbb{R} \) such that \( f_1 - f_2 / -2 \equiv C \), hence \( f_2 \sim f_1 \)

(iii). Transitive: If \( f_1 \sim f_2 \) and \( f_2 \sim f_3 \), then there exists \( C_1 \in \mathbb{R} \) and \( C_2 \in \mathbb{R} \) such that \( f_1 - f_2 \equiv C_1 \) and \( f_2 - f_3 \equiv C_2 \), so \( f_1 - f_3 = (f_1 - f_2) + (f_2 - f_3) \equiv C_1 + C_2 \), hence \( f_1 \sim f_3 \)

6. If \( R \) and \( R' \) are two equivalence relations on a set \( S \), is \( R \cap R' \) also an equivalence relation on \( S \)?

**Solution:** \( R \cap R' \) is an equivalence relation.

(i). Reflexive: for any \( x \in S \), \( (x, x) \in R \) and \( (x, x) \in R' \), so \( (x, x) \in R \cap R' \)

(ii). Symmetric: If \( (x, y) \in R \cap R' \), then \( (x, y) \in R \) and \( (x, y) \in R' \). \( R \) and \( R' \) are equivalence relations, so \( (y, x) \in R \) and \( (y, x) \in R' \), hence \( (y, x) \in R \cap R' \)

(iii). Transitive: If \( (x, y) \in R \cap R' \) and \( (y, z) \in R \cap R' \), then \( (x, y) \in R \) and \( (y, z) \in R \) implies \( (x, z) \in R \), and \( (x, y) \in R' \) and \( (y, z) \in R' \) implies \( (x, z) \in R' \), we get \( (x, z) \in R \cap R' \)

7. If \( G \) is a group of order \( p^n \), where \( p \) is a prime and \( n > 1 \). Prove \( G \) contains an element of order \( p \).

**Solution:** Pick any non-identity element \( x \in G \). \( |x| \) divides \( |G| = p^n \), so \( |x| = p^r \) for some \( 1 \leq r \leq n \).

If \( r = 1 \), then \( x \) is an element of order \( p \), done.

If \( r > 1 \), consider the element \( y = x^{p^{r-1}} \): \( y^p = (x^{p^{r-1}})^p = x^{p^r} = 1 \), so \( |y| = p \) if we can show \( y \neq 1 \).

Suppose \( y = 1 \), then this is to say \( x^{p^{r-1}} = 1 \), contradict to \( |x| = p^r \), we conclude that \( y \neq 1 \).

8. If \( G \) has five subgroups of order 7, prove \( G \) has at least 35 elements.

**Solution:** If \( H \) and \( K \) are two subgroups of order 7 in \( G \) such that \( H \neq K \), then \( H \cap K = \{1\} \). Suppose there is non-identity \( x \in H \cap K \), then in particular, \( x \in H \), so \( |x| \) divides \( |H| = 7 \), and \( x \) is not the identity, it follows \( |x| = 7 \), so \(< x > = H \). Similarly, \(< x > = K \), we get \( H = K \), contradict to the assumption \( H \neq K \), therefore such \( x \) does not exists. we conclude \( H \cap K = \{1\} \).
If there are five different subgroups of order 7, then by the above paragraph, except the identity element, any two of those five subgroups share no element. It follows there are $1 + 5 \times (7 - 1) = 31$ elements in the union of these five subsets, so $|G| \geq 31$.

$G$ contains subgroups of order 7 implies 7 divides $|G|$, and 35 is the smallest multiple of 7 that is no smaller than 31, so we conclude $|G| \geq 35$. 

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