Definition. A partition $\mathcal{P}$ of a set $S$ is a subdivision of $S$ into nonoverlapping, nonempty subsets:

$$S = \bigcup_{i \in I} S_i,$$

where $I$ is some index.

Example. We can make a partition of $\mathbb{Z}$ into the set of odd numbers and the set of even numbers.

Definition. Given a set $S$, a relation on $S$ is a subset $R \subseteq S \times S$, that is, a subset of the ordered pairs of elements in $S$.

Definition. A relation $R$ on $S$ is called an equivalence relation if it satisfies:

1. (Reflexive): $\forall x \in S, (x, x) \in R$
2. (Symmetric): $((x, y) \in R) \Rightarrow (y, x) \in R$
3. (Transitive): $((x, y) \in R, (y, z) \in R) \Rightarrow (x, z) \in R$

Notation. When a relation is an equivalence relation, we usually write $x \sim y$ if $(x, y) \in R$.

Example. On the set of integers $\mathbb{Z}$, define $x \sim y$ if $x - y \in 2\mathbb{Z}$ this is an equivalence relation:

1. $\forall k \in \mathbb{Z}, k - k = 0$ is even
2. If $k_1 - k_2 \in 2\mathbb{Z}$, then $k_2 - k_1 \in 2\mathbb{Z}$.
3. If $k_1 - k_3 \in 2\mathbb{Z}$ and $k_2 - k_3 \in 2\mathbb{Z}$, then $k_1 - k_3 - (k_1 - k_2) = (k_2 - k_3) \in 2\mathbb{Z}$.
Example. \[ X = \{ f : [0, 1] \to \mathbb{R} | f \text{ is integrable} \} \]

Define \( f_1 \sim f_2 \) if \( \int_0^1 |f_1 - f_2| \, dx = 0 \).

This is an equivalence relation:

(i) \( \forall f \in X, \int_0^1 |f - 0| \, dx = 0 \)

(ii) If \( \int_0^1 |f_1 - f_2| \, dx = 0 \), then \( \int_0^1 |f_1 - f_3| \, dx = 0 \)

(iii) If \( \int_0^1 |f_1 - f_2| \, dx = 0 \) and \( \int_0^1 |f_2 - f_3| \, dx = 0 \), then \( \int_0^1 |f_1 - f_3| \, dx = 0 \).

Definition. Given an equivalence relation on a set \( S \), define the equivalence class of \( a \in S \) to be the subset \( [a] = \{ b \in S | a \sim b \} \).

Proposition. Given an equivalence relation on a set \( S \), \( a, b \in S \). Then either \( [a] = [b] \) or \( [a] \cap [b] = \emptyset \).

Proof. Suppose \( [a] \cap [b] \neq \emptyset \), then \( \exists x \in [a] \cap [b] \).

\( x \in [a] \cap [b] \Rightarrow a \sim x \) and \( b \sim x \).

\( \forall y \in [a], a \sim y \Rightarrow y \sim a \sim x \sim b \Rightarrow y \in [b] \).

So \( [a] \subseteq [b] \).

Similarly, \( [b] \subseteq [a] \).

So \( [a] = [b] \).

Proposition. The equivalence classes of an equivalence relation on \( S \) give a partition of \( S \), and conversely a partition of \( S \) defines an equivalence relation on \( S \).
Proof. We have just proved that different equivalence classes are disjoint, and their union will be the set $S$ since \( \forall x \in S, \ x \in C_x \).
So $S$ is the disjoint union of all its equivalence classes.

Conversely, if there is a partition on $S$, we can define $a \sim b$ if $a$ & $b$ lie in the same part of the partition, and it’s not hard to check this is an equivalence relation.

Definition. $S$ is a set with an equivalence relation defined on it.
Let $\overline{S}$ be the set of all equivalence classes.
Each equivalence class can be also written as $[a] = \overline{a}$.
There is a map $\pi: S \rightarrow \overline{S}$
$x \rightarrow \overline{x}$.
which sends each element in $S$ to its equivalence class.

Example. Recall that we have defined an equivalence relation on $\mathbb{Z}$
that $x \sim y$ if $x - y \in 2\mathbb{Z}$, i.e. $x - y$ is even.
There are two distinct equivalence classes, $\overline{0}$ and $\overline{1}$.
$\overline{0}$ is the set of even numbers, and
$\overline{1}$ is the set of odd numbers.