Topologically protected states in one-dimensional continuous systems and Dirac points

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We study a class of periodic Schrödinger operators on \(\mathbb{R}\) that have Dirac points. The introduction of an “edge” via adiabatic modulation of a periodic potential by a domain wall results in the bifurcation of spatially localized “edge states,” associated with the topologically protected zero-energy mode of an asymptotic one-dimensional Dirac operator. The bound states we construct can be realized as highly robust transverse-magnetic electromagnetic modes for a class of photonic waveguides with a phase defect. Our model captures many aspects of the phenomenon of topologically protected edge states for 2D bulk structures such as the honeycomb structure of graphene.

Floquet–Bloch Theory

Let \(Q \in C^0\) denote a one-periodic real-valued potential, i.e., \(Q(x+1) = Q(x), \ x \in \mathbb{R}\), and consider the Schrödinger operator

\[
H_Q = -\partial_x^2 + Q(x).
\]

**Definition 1:** The space of \(k \in \text{ pseudoperiodic } L^2\) functions is given by

\[
L^2_k = \{ f \in L^2_{\text{loc}} : f(x+1;k) = e^{ik}f(x;k) \}.
\]

The Sobolev spaces \(H^N_{L^2} \; , \; N = 0, 1, \ldots \) are analogously defined. Because the \(k \in \text{ pseudoperiodic }\) boundary condition is invariant under \(k \rightarrow k+2\pi\), it is natural to work with a fundamental dual-period cell or Brillouin zone, which we take to be \(B = [0, 2\pi]\).

We next consider a one-parameter family of Floquet–Bloch eigenvalue problems, parameterized by \(k \in B\):

\[
H_Q\Phi(E) = E\Phi, \quad \Phi(x+1;k) = e^{ik}\Phi(x;k). \tag{1}
\]

The eigenvalue problem [1] is self-adjoint on \(L^2_k\) and has a discrete set of eigenvalues, listed with repetitions

\[
E_1(k) \leq E_2(k) \leq \cdots \leq E_j(k) \leq \cdots
\]

with corresponding \(L^2_k\) eigenfunctions \(\Phi_1(x;k), \ \Phi_2(x;k), \ldots\), which, for fixed \(k\), can be taken to be a complete orthonormal set in

**:Significance**

Topological insulators (TIs) have been a topic of intense study in recent years. When appropriately interfaced with other structures, TIs possess robust edge states, which persist in the presence of localized interface perturbations. Therefore, TIs are ideal for the transfer or storage of energy or information. The prevalent analyses of TIs involve idealized discrete tight-binding models. We present a rigorous study of a class of continuum models, for which we prove the emergence of topologically protected edge states. These states are bifurcations at linear band crossings (Dirac points) of localized modes. The bifurcation is induced by the \(0\)-energy eigemnode of a class of one-dimensional Dirac equations.

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Furthermore, the states $\Phi_j(x; k)$, $j \geq 1$, $k \in \mathcal{B}$ are complete in $L^2(\mathbb{R})$,

$$f(x) \in L^2(\mathbb{R}) \Rightarrow f(x) = \sum_{j \geq 1} \hat{f}_j(k) \Phi_j(x; k) dk,$$

where $\hat{f}_j(k) = (\Phi_j(\cdot; k), f)_{L^2(\mathbb{R})}$.

We make use of the decompositions $L^2_k = L^2_{k,e} \oplus L^2_{k,o}$ and $H^j_k = H^j_{k,e} \oplus H^j_{k,o}$ into subspaces defined in terms of even- and odd-index Fourier projections, introduced as follows:

**Definition 2:**

1) $L^2_{k,e}$ is the subspace of $L^2_k$ consisting of functions of the form

$$e^{i k x} P_{a}(x) = e^{i k x} \sum_{m \in \mathbb{Z}} P(m) e^{2\pi i m x}, \quad \sum_{m \in \mathbb{Z}} |P(m)|^2 < \infty;$$

i.e., $P_a(x)$ is an even-index 1-periodic Fourier series.

2) $L^2_{k,o}$ is the subspace of $L^2_k$ consisting of functions of the form

$$e^{i k x} P_o(x) = e^{i k x} \sum_{m \in \mathbb{Z} + 1} P(m) e^{2\pi i m x}, \quad \sum_{m \in \mathbb{Z} + 1} |P(m)|^2 < \infty;$$

i.e., $P_o(x)$ is an odd-index 1-periodic Fourier series.

3) Sobolev spaces, $H^m_{k,e}$ and $H^m_{k,o}$, $m = 0, 1, 2, \ldots$, are defined in the natural way.

**Motivating Example**

Start with a smooth, real-valued, even, and 1-periodic function, $Q(x)$. Define the one-parameter family of potentials $Q(x; s)$, a sum of translates of $Q(x)$:

$$Q(x; s) = Q(x + s) + Q(x - s), \quad 0 \leq s \leq 1.$$  

Clearly, $Q(x; s)$ is 1-periodic. Moreover, $Q(x; s = 1/2)$ has minimal period equal to $1/2$. That is, $Q(x; s)$ has an additional translation symmetry at $s = 1/2$ (Fig. 1). The function $Q(x; s)$ may be expressed as a Fourier series

$$Q(x; s) = \sum_{m \in \mathbb{Z}_s} Q_m \cos(\pi m s) \cos(2\pi m x),$$

which for $s = 1/2$ reduces to an even-index cosine series:

$$V_c(x) = Q(x; 1/2) \equiv \sum_{m \in \mathbb{Z}_s} \tilde{Q}_m \cos(2\pi m x). \quad [2]$$

**Dirac Points**

The operator $-\partial_x^2 + V_c(x)$ can be shown to have distinguished (“symmetry-protected”) quasi-momentum/energy pairs, called Dirac points, at which neighboring spectral bands touch and at which dispersion loci cross linearly (Fig. 2).

**Theorem 1.** Consider the Schrödinger operator $H = -\partial_x^2 + V_c(x)$, where $V_c(x)$ is a generic smooth even-index cosine series. Then, $H$ has Dirac points $(k_0, \pm E_0, E_0)$ in the following sense:

1. There exists $b_0 \geq 1$ such that $E_0 = E_{b_0}(\pi) = E_{b_0+1}(\pi)$.
2. $E_0$ is an $L^2_k$ eigenvalue of multiplicity 2.
3. $H^2_{k,e} = H^2_{k,e} \oplus H^2_{k,o}$, where $H : H^2_{k,e} \to L^2_{k,e}$ and $H : H^2_{k,o} \to L^2_{k,o}$.
4. The inversion $S[f](x) = f(-x)$, where $S$ commutes with $H(k_0)$ and $S[H_{k,e}(\cdot)] = H_{k,e}(\cdot)$, $S[H_{k,o}(\cdot)] = H_{k,o}(\cdot)$, is such that $S[H_{k,e}(\cdot)] = S[H_{k,o}(\cdot)] = S[H_{k_o}(\cdot)]$.
5. The $L^2_k$ null-space of $H - E_0$ is spanned by functions $\Phi \in L^2_{k,e}$, $\Phi_c \equiv S[\Phi](x) \in L^2_{k,o}$, $(\Phi_1, \Phi_2, \Phi_3) = (0, 1, 2)$, $a, b, c = 0, 1, 2$.
6. The “Fermi velocity” satisfies

$$\lambda_0 = 2i(\Phi_1, i\Phi_2)_L = \delta_{ab} \neq 0 \quad [3]$$

(for generic $V_c$), and there exist $\zeta_0 > 0$ and Floquet-Bloch eigenpairs

$$(\Phi_1(x; k), E_c(x) \frac{\partial}{\partial x}) \quad (\Phi_o(x; k), E_o(x) \frac{\partial}{\partial x})$$

and smooth functions $\eta_\pm(x)$, with $\eta_\pm(0) = 0$, defined for $|k - k_0| < \zeta_0$ and such that $\Phi(x; k)$ have the same span as the functions $\Phi(x; k)$.

**Fig. 2.** Left illustrates the existence of Dirac points for a periodic potential, $V_c$ of the type plotted in Fig. 3. Top. Superimposed on $V_c$ is the mode $\Phi_1 \in L^2_{k,e}$.

In ref. 24, Dirac points play a central role in the construction of one-dimensional almost periodic potentials for which the Schrödinger operator has nowhere a dense spectrum.

**Dimers**

For $s \neq 1/2$, $Q(x; s)$ is a periodic potential consisting of dimers, double-well potentials in each period cell (Fig. 1). Set $s^d = 1/2 + \delta s_0$, with $0 < \delta < 1$ and $0 \neq s_0 \in \mathbb{R}$ fixed. Then, $Q(x; s^d)$ is of the form

$$Q(x; s^d) = \sum_{m \in \mathbb{Z}_s} Q_m^d \cos(2\pi m x) + \delta s_0 \sum_{m \in \mathbb{Z}_s + 1} W_m^d \cos(2\pi m x),$$

where $Q_m^d, W^d_m = O(1)$ as $\delta \to 0$. The operator

$$H \left( \frac{1}{2} + \frac{\delta s_0}{\partial x} \right) = -\partial_x^2 + Q(x; s^d)$$

is a Hill’s operator (25). The character of its spectrum is well known (Fig. 4, Upper). For \( \delta \) fixed, the gaps that open at the Dirac points have widths of order $O(\delta)$.

Now fix a constant $\kappa_\infty > 0$ and let $s^d(x) = 1/2 + \delta x$.

We assume that $\kappa(X)$ is sufficiently smooth and approaches its asymptotic values sufficiently rapidly. The operator $H(s^d(x)) = \cdots$
Consider the eigenvalue problem for the Schrödinger operator $H_0$. Let $(k_x, x, E)$ denote a Dirac point of $-\beta^2 + V_c(x)$ in the sense of Theorem 1. Furthermore, assume the condition

$$\theta_\delta = (\Phi_1, W_0, \Phi_2)_{L^2(\mathbb{R}^2)} \neq 0,$$

which holds for generic $V_c$ and $W_0$. Assume that $k(X) \to \pm \kappa_{\infty}$ sufficiently rapidly as $X \to \pm \infty$. Then, the following holds:

1) There exists $\delta_0 > 0$ and a branch of solutions to the eigenvalue problem $H_0 \psi^0 = \epsilon^0 \psi^0$

where $\epsilon^0 = \epsilon^0(\delta)$ bifurcating from energy $E_c$ at $\delta = 0$, into the gap $I_\delta$ (Proposition 2), with $|E^0 - E_c| \leq \delta^2$, and with corresponding spatially localized (exponentially decaying) eigenstate, $\psi^0$.

2) $\psi^0(x)$ is approximated by a slowly varying and spatially decaying modulation of the degenerate Floquet–Bloch modes $\Phi_1$ and $\Phi_2$:

$$|\psi^0(x) - \delta^{1/2} [\alpha_1(\delta \xi) \Phi_1(x) + \alpha_2(\delta \xi) \Phi_2(x)]|_{H_0(I_\delta)} \leq \delta.$$

3) The amplitudes, $\alpha(X) = (\alpha_1(X), \alpha_2(X))$, are governed by the topologically protected zero eigenmode of the $2 \times 2$ matrix Dirac operator,

$$D \equiv \i \lambda_0 \sigma_3 \partial_X + \theta_\delta \kappa(X) \sigma_1,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\lambda_0 \theta_\delta \neq 0$ (Eqs. 3 and 4).

Theorem 3 is illustrated in Figs. 2–4. Fig. 2, Right displays the midgap eigenvalues, $E^0$. Fig. 3 displays the unperturbed

- $\delta^2 + Q(x, s^0(x))$ is the Hamiltonian for a structure that adiabatically transitions between dimer periodic structures at $\pm \infty$ through a domain wall. For $\delta$ small,

$$Q(x, s^0(x)) \approx V_c(x) + \delta \kappa(\delta \xi) W_0(x),$$

where $V_c(x)$ denotes an even-index cosine series, a structure with a Dirac point, and $W_0(x)$, an odd-index cosine series, is a noncompact perturbation induced by the domain wall (phase defect).

This motivates our study of the family of operators:

$$H_\delta = -\beta^2 + V_c(x) + \delta \kappa(\delta \xi) W_0(x).$$

The operator $H_\delta$ interpolates adiabatically, through a domain wall, between the operators

$$H\left(\frac{1}{2} - \kappa_{\infty} \delta\right) \equiv H_{\delta, -} \equiv H\left(\frac{1}{2}\right) - \delta \kappa_{\infty} W_0 \text{ at } x = -\infty,$$

$$H\left(\frac{1}{2} + \kappa_{\infty} \delta\right) \equiv H_{\delta, +} \equiv H\left(\frac{1}{2}\right) + \delta \kappa_{\infty} W_0 \text{ at } x = +\infty.$$

The noncompact perturbations $\delta \kappa(\delta \xi) W_0(x)$ and $\pm \delta \kappa_{\infty} W_0(x)$, which break the symmetries of $V_c(x)$, open a gap in the essential spectrum.

Proposition 2. Let $(k_x, E)$ denote a Dirac point of $-\beta^2 + V_c(x)$. Assume the condition $\theta_\delta = (\Phi_1, W_0, \Phi_2)_{L^2(\mathbb{R}^2)} \neq 0$, which holds for generic $W_0$ and $V_c$. Fix $c$ less than but arbitrarily close to 1 and define the real interval

$$I_\delta \equiv (c \kappa_{\infty} |\theta_\delta|, c \kappa_{\infty} |\theta_\delta|).$$

Then, there exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$

$$I_\delta \cap \sigma_{\text{ess}}(H_\delta), I_\delta \cap \sigma_{\text{ess}}(H_{\delta, -}), \text{ and } I_\delta \cap \sigma_{\text{ess}}(H_{\delta, +})$$

are all empty sets.
potential, $V_e(x)$, and the mode $\Phi_i(x)$ (Fig. 3, Top), the domain-wall modulated potential $V_{i}(x) + \delta(x) W_{i}(x)$ (Fig. 3, Middle), and the midgap mode $\Psi^0(x)$ (Fig. 3, Bottom). Fig. 4, Lower displays the bifurcation curves of topologically protected states.

**Remark 1:**

1) The zero mode of a one-dimensional Dirac operator, $\mathcal{D}$, plays an important role in refs. 3, 21, and 22.

2) The bifurcation discussed in Theorem 3 is associated with a noncompact perturbation of $H_0 = -\partial_x^2 + V_e(x)$, namely a phase defect across the structure, which at once changes the essential spectrum and spawns a bound state. This is in contrast to bifurcations from the edge of continuous spectra, arising from localized perturbations (for example, refs. 26–31). A class of edge bifurcations due to a noncompact perturbation is studied in ref. 32.

3) Our model captures many aspects of the phenomenon of topologically protected edge states for 2D bulk structures such as the honeycomb structure of graphene (for example, refs. 3 and 13).

**Proof of Theorem 1**

To establish that $(k_0, E_0)$ is a Dirac point, we verify the sufficient conditions of the following:

**Theorem 4.** Consider $H = -\partial_x^2 + V_e$, where $V_e$, given by Eq. 2, is sufficiently smooth. Let $k_0 = \pi$ and assume that $E_0$ is a double eigenvalue, lying at the intersection of the $b^h$ and $(b^h + 1)^h$ spectral bands:

$$E_0 = E_{b^h}(k_0) = E_{b^h+1}(k_0).$$

Assume the following conditions:

I) $E_0$ is a simple $L_{k_0,e}^2$ eigenvalue of $H$ with one-dimensional eigenspace

$$\text{span} \{ \Phi_1(x) \} \subset L_{k_0,e}^2.$$  

II) $E_e$ is a simple $L_{k_0,o}^2$ eigenvalue of $H$ with one-dimensional eigenspace

$$\text{span} \{ \Phi_2(x) = \Phi_1(x) \} \subset L_{k_0,o}^2.$$  

III) Nondegeneracy condition

$$0 \neq \lambda_0 \equiv 2\langle \Phi_1, \partial_x \Phi_1 \rangle = -2\pi \left\{ 2 \sum_{m \in \mathbb{Z}} m |c_1(m)|^2 + 1 \right\}.$$  

Here, $\{ c_1(m) \}_{m \in \mathbb{Z}}$ denote the $L_{k_0,e}^2$ Fourier coefficients of $\Phi_1(x)$. Then, $(k_0, E_0)$ is a Dirac point in the sense of Theorem 1.

The proof follows that of theorem 4.1 of ref. 13 for the case of honeycomb lattice potentials in $\mathbb{R}^2$. To establish Theorem 1, that $-\partial_x^2 + V_e$ has Dirac points for generic $V_e$, we consider the family of operators $H'(e) = -\partial_x^2 + eV_e$. The conditions of Theorem 4 can be verified for all $e \in [0, \epsilon_0)$, for some $\epsilon_0(E_0) > 0$ by a perturbative/Lyapunov–Schmidt reduction argument about quasi-momentum energy pairs for $e = 0$: $(k_0, E_{m_0}) = (\pi, (2m + 1)^2 \pi^2)$, $m = 0, 1, \ldots$. A continuation argument to $e$ of arbitrary size is implemented following the strategy of ref. 13. This shows the persistence of the conditions of Theorem 4 and therefore the existence of a Dirac point, $(k_0, E_{k_0}^{(e)})$, for all $e \not\in \mathbb{C}$, where $\mathbb{C}$ denotes a countable and closed subset of $\mathbb{R}$.

**Proof of Theorem 3**

A formal multiple-scale expansion anticipates the form of the solution at any finite order in $\delta$. To establish the validity of this expansion, the corrector is decomposed into its “near-energy” and “far-energy” components, using the spectral decomposition of the unperturbed operator: $-\partial_x^2 + V_e$. The near-energy regime corresponds to energies within the intersecting spectral bands, which are near $E_0$ and quasi-momentum, $k$, satisfying $|k - k_0| \leq \delta$, where $0 < \delta \ll 1$. The far-energy components correspond to all other energies (within the intersecting bands and all other bands). The eigenvalue problem is equivalent to a coupled infinite system for the near- and far-energy spectral components. To solve this system we first solve for the far-energy components as a functional of the near-energy components. This leads to a Lyapunov–Schmidt reduction to a closed nonlocal system, which determines the near-energy components of the corrector and the correction to the degenerate eigenvalue. Under rescaling, the latter equation may be written as a Dirac-type equation with Dirac operator $D$ (Eq. 5) bound limited to rescaled momenta $|\xi| \leq \delta^{-1}$. We then solve this system for all $\delta$ positive and sufficiently small.

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