Radar Cross Section Reduction of a Cavity in the Ground Plane: TE Polarization

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Abstract. The reduction of backscatter radar cross section (RCS) in TE polarization for a rectangular cavity embedded in the ground plane is investigated in this paper. It is established by placing a thin, multilayered radar absorbing material (RAM) with possibly different permittivities at the bottom of the cavity. A minimization problem with respect to the backscatter RCS is formulated to determine the synthesis of RAM. The underlying scattered field is governed by a generalized Helmholtz equation with transparent boundary condition. The gradient with respect to the material permittivity is derived by the adjoint state method. A fast solver for the Helmholtz equation is presented for the optimization scheme. Numerical examples are presented to show the efficiency of the algorithm for RCS reduction.

1. Introduction. Radar cross section (RCS) is an important measure for radar system to characterize the scattering from a target [19]. Reducing the RCS for a cavity structure is of great interest for civil and military applications due to its broad existence in practice, examples including jet engine inlet ducts, vehicles with grooves and cavity-backed antennas. One of the effective approaches for RCS reduction is to use multilayered radar absorbing material (RAM), which is presumably thin and light, to absorb the energy. The mechanism underlying the RAM is to convert electromagnetic energy into heat and consequently the reflected energy is reduced. Our aim in this paper is to study the optimal design of a cavity with multilayered RAM, so that the RCS is reduced over a range of incident angles or frequencies.

More specifically, this paper focuses on the backscatter RCS reduction for a cavity embedded in the infinite ground plane. The cavity is assumed to be invariant in the $z$ direction (Figure 1) and illuminated by a time harmonic plane wave. A thin, multilayered RAM is coated at the bottom of the cavity. Assume the material has the same magnetic permeability as the empty space, but different permittivities at

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each layer. The permittivity for each layer is determined through the optimization
that minimizes the backscatter RCS. Based on the coevaporation technique\cite{18},
the admissible set for the permittivity is assumed to be a continuous variable in a
specified interval.

Lots of optimization techniques are proposed to reduce the RCS for an object
with multilayered RAM \cite{12, 23, 25}. When the material is chosen from a discrete
set, genetic algorithm(GA) is considered as a practical optimization method, since
numerically it leads to an output that is close to the global minimum. However, GA
is a stochastic based method and generally requires a large amount of simulations
to achieve convergence. For our problem, the admissible set for the material is not
discrete. Meanwhile, for a given material, solving the forward scattering problem
needs a considerable amount of time. Therefore, it is inappropriate to apply GA
to the optimal design problem. Instead, a gradient based approach is desired. In
particular, the method we employ is sequential quadratic programming(SQP), which
is effective for nonlinear optimization problem with continuous design variables \cite{24}.
Although the gradient based method generally leads to a local minimum, numerical
experiments show these local minimums are satisfactory with a good initial guess.

As one of the essential components in our optimization method, the gradient for
the RCS with respect to the permeability is evaluated through the adjoint state
method. The adjoint state method is used widely in the nonlinear optimization
problem especially when the state variable is taken from the solution of a differential
equation \cite{13}. Although the gradient can also be simply evaluated by numerical
difference, the accuracy will be lost given the sensitivity of the problem. Meanwhile,
the time to evaluate the numerical gradient depends linearly on the number of layers,
while the gradient based on the adjoint state method is independent of the amount
of layers.

Since our problem is essentially two dimensional, there exist two fundamental
polarizations for Maxwell’s equation: TM (Transverse magnetic) polarization and
TE (Transverse electric) polarization. This paper focuses on the TE polarization,
which continues our study for TM polarization in \cite{7}. Two difficulties arise from
TE case compared to TM case. The first is that TE polarization is governed by a
generalized Helmholtz equation, where the permittivity is included in a differential
operator. The derivation for the gradient via the adjoint state method would be
more involved. The second is the fast algorithm used in \cite{8} for the forward scattering
problem does not directly apply to TE polarization when the material has layered
structure. Both of these issues will be addressed in this paper. In particular,
we extend the algorithm proposed in \cite{8} to the layered medium for TE case by
employing the continuity condition. The resulted algorithm is fast and accurate,
which exactly fits the needs for an optimization algorithm.

It is worth mentioning that the cavity problem has been extensively studied in
terms of both mathematical analysis and numerical simulation. In \cite{1, 2, 3}, the
existence and uniqueness for the forward scattering problem of a cavity were es-
lished in a rather general setting. The inverse scattering problem was studied
in \cite{5}, where the uniqueness and the stability results were investigated. In \cite{9},
the authors considered the scattered field of a rectangular cavity illuminated by a
high frequency wave. The explicit dependence on the wavenumber for the energy
at the aperture were given. Numerical simulation to evaluate the RCS for large
and deep cavities is considered as a great challenge over the last decades. Lots of
numerical methods have been proposed to this problem\cite{4, 10}. Standard methods
include finite difference[8], finite element with boundary integral[16, 21], method of momentum[15]. High frequency techniques include Gaussian beam asymptotics[11], shooting and bouncing rays(SBR)[20], iterative physical optics[14]. A comprehensive review for all the computational techniques developed over the last two decades can be found in [4]. Recently, we developed a fast mode matching method for three dimensional rectangular cavity in [6].

The paper is organized as follows. In the next section, the optimal design of a cavity with multilayered RAM is formulated as a minimization problem, with the constraint as a generalized Helmholtz equation with artificial boundary condition. The objective function and design variables are also defined in this section. The gradient based on the adjoint state method is derived in Section 3, following a remark on the finite dimensional counterpart. Section 4 gives a brief introduction on the SQP optimization technique followed by a description of the minimization algorithm. The solver for the forward scattering problem is presented in Section 5. Numerical experiments are provided in Section 6 to show the efficiency of the method on RCS reduction. Section 7 concludes the whole discussion.

2. Problem formulation. Consider a rectangular cavity \( \Omega = [0,a] \times [-b, 0] \) embedded in the ground plane illuminated by a time harmonic plane wave, as illustrated in Figure 1. The problem is formulated in 2-D by assuming that the cavity and the material are invariant in the \( z \) direction. Above the ground plane, the medium is assumed to be homogeneous with dielectric permittivity \( \varepsilon_0 \). The surface of the ground plane \( \Gamma^c \) and the boundary \( S \) of the cavity are assumed to be perfect conductors. The bottom of the cavity is coated by a thin, layered medium with permittivities given by \( \varepsilon_j, j = 1, 2, \cdots, L \), where \( L \) is the number of layers. Above the coating material, the cavity is filled with the same material as the upper half space. For convenience, denote \( \varepsilon \) as the permittivity function in terms of variable \( y \) and \( k = \omega \sqrt{\varepsilon \mu_0} \) as the wavenumber in the whole space. In this paper, we consider TE(transverse electric) polarization and assume the magnetic permeability \( \mu_0 \) is the same everywhere. The goal is to determine the permittivities of the coating material within a presumably small thickness \( h \), such that the backscatter RCS is minimized.

Assume the incident field is given by a plane wave: \( u^i = e^{i(\alpha x - \beta y)}, \) where \( \alpha = k_0 \cos \theta, \beta = k_0 \sin \theta, \theta \) is the incident angle with respect to the \( x \)-axis and \( k_0 = \)
\[ \omega \sqrt{\varepsilon_0 \mu_0} \] is the wavenumber in the upper half space \( \mathbb{R}^{2+} \). Denote \( u(x, y) \) and \( u^s(x, y) \) as the total field and the scattered field respectively. Following the formulation in [17], the total field \( u(x, y) \) is the summation:

\[
u(x, y) = u^s(x, y) + u^r + e^{i(\alpha x + \beta y)}
\]

where \( e^{i(\alpha x + \beta y)} \) is identified as the reflected field \( u^r \). \( u^s(x, y) \) satisfies the following generalized Helmholtz equation, which can be easily derived from the time harmonic Maxwell equation:

\[
\begin{cases}
\nabla \cdot \left( \frac{1}{k^2} \nabla u \right) + u = 0 & \text{in } \Omega \cup \mathbb{R}^2^+ \\
\frac{\partial u}{\partial n} = 0 & \text{on } \Gamma^c \cup S
\end{cases}
\]

We now apply Fourier Transform method to derive a transparent boundary condition on \( \Gamma \), by which Eq (2) is reduced to a bounded domain problem. Ideas of transparent boundary condition have been applied broadly in many wave propagation problems. One resorts to [22] and reference therein for the applications.

Let \( H^1_{loc}(\mathbb{R}^{2+}) \) be the Sobolev space for functions that are locally integrable in \( \mathbb{R}^{2+} \). The exterior scattered field \( u^s(x, y) \in H^1_{loc}(\mathbb{R}^{2+}) \) satisfies the equation:

\[
\begin{cases}
\Delta u^s + k_0^2 u^s = 0 & \text{in } \mathbb{R}^{2+} \\
\frac{\partial u^s}{\partial n} = \phi(x) & \text{on } \Gamma \\
\frac{\partial u^s}{\partial n} = 0 & \text{on } \Gamma^c
\end{cases}
\]

Together with the radiation condition:

\[
\lim_{\rho \to \infty} \rho^{1/2} \left( \frac{\partial u^s}{\partial \rho} - ik_0 u^s \right) = 0
\]

Here \( \rho = \sqrt{x^2 + y^2} \) and \( \phi(x) \) is assumed to be known. Let \( \tilde{\phi}(x) \) denote the zero extension of \( \phi(x) \) on \( y = 0 \). Taking a Fourier Transform for \( u^s \) with respect to \( x \) and denoting it as \( \hat{u^s}(\xi, y) \), we have:

\[
\begin{cases}
\frac{\partial^2 \hat{u^s}}{\partial y^2} + (k^2 - \xi^2) \hat{u^s} = 0 & \text{for } y > 0, \\
\frac{\partial \hat{u^s}}{\partial n} = \tilde{\phi}(\xi) & \text{on } y = 0.
\end{cases}
\]

Using the outgoing radiation condition (4), we easily obtain the solution for (5):

\[
\hat{u^s}(\xi, y) = \frac{\tilde{\phi}(\xi)}{i\sqrt{k_0^2 - \xi^2}} e^{i\sqrt{k_0^2 - \xi^2} y}, \text{ if } \xi \neq k_0
\]

Taking an inverse Fourier Transform with respect to \( \xi \) and letting \( y = 0 \) yields:

\[
u^s(x, 0) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{\sqrt{k_0^2 - \xi^2}} \tilde{\phi}(\xi) e^{i\xi x} d\xi
\]

Observing that \( \phi(x) = \frac{\partial u^s}{\partial n} = \frac{\partial u}{\partial n} \) on \( \Gamma \), we therefore obtain the transparent boundary condition for \( u \):

\[
u(x, 0) = T \left( \frac{\partial u}{\partial n} \right) + g, \text{ on } \Gamma
\]

where

\[
T \left( \frac{\partial u}{\partial n} \right) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{\sqrt{k_0^2 - \xi^2}} \frac{\partial \tilde{u}}{\partial n} e^{i\xi x} d\xi,
\]

\[
g = 2e^{i\alpha x},
\]
and \(\widetilde{\partial u/\partial n}\) denotes the zero extension of \(\partial u/\partial n\) on \(y = 0\).

Equipped with the transparent boundary condition, Eq. (2) is reduced to a bounded domain problem:

\[
\begin{cases}
\nabla \cdot \left( \frac{1}{\varepsilon} \nabla u \right) + u = 0, & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0, & \text{on } S \\
u(x, 0) = T \left( \frac{\partial u}{\partial n} \right) + g, & \text{on } \Gamma
\end{cases}
\] (9)

Eq. (9) is uniquely solvable for \(u \in H^1_g(\Omega)\), where

\[
H^1_g(\Omega) = \left\{ \phi \in H^1(\Omega) \mid \frac{\partial \phi}{\partial n} \in \tilde{H}^{-1/2}(\Gamma), \phi = T \left( \frac{\partial \phi}{\partial n} \right) + g \text{ on } \Gamma \right\},
\] (10)

and \(\tilde{H}^{-1/2}(\Gamma)\) is the dual space of \(H^{1/2}(\Gamma)\).

The proof for the existence can be found in [2]. The normal derivative of \(u\) at the aperture \(\Gamma\) can be found through the numerical solution of (9) and then used to evaluate the backscatter RCS. The formula of RCS is given in the next section.

### 2.1. Objective function.

This paper aims to reduce the backscatter RCS, since in general, larger RCS of an object indicates easier detection by the radar. The definition of RCS in the two dimension is given by:

\[
\sigma(\varphi, \theta) = \lim_{\rho \to \infty} \frac{2 \pi \rho |u_s(\varphi, \theta)|^2}{|u_i(\theta)|^2},
\] (11)

where \(\theta\) is the incident angle and \(\varphi\) is the observation angle. When \(\varphi = \theta\), it is called backscatter RCS or monostatic RCS. The reason we consider reducing backscatter RCS only is because most radar systems are monostatic in practice[19].

Since the upper half space is homogeneous, the backscatter RCS of the cavity in TE polarization can also be evaluated through\[17\]:

\[
\sigma := \frac{4}{k_0} \left| \int_{\Gamma} \frac{1}{2} \frac{\partial u}{\partial n} e^{-ik_0x \cos \theta} dx \right|^2
\] (12)

which is defined as the objective function in this paper. If the RCS reduction over a set of incident angles or frequencies is desired, the cost function may be changed to the following:

\[
\sigma_{total} := \sum_{l=1}^{r} w_l \sigma(t_l)
\] (13)

where \(t_l\) represents the dependence on different angles or frequencies, and \(w_l > 0\), \(l = 1, 2, \cdots, r\) are the corresponding weights. Furthermore, depending on which angle is important, these weights may be chosen accordingly.

Since the permittivity \(\varepsilon\) is the design variable and \(u\) depends on \(\varepsilon\) from the model (9), the optimization problem can be rewritten as:

\[
\min_{\varepsilon \in \Lambda} \sigma(\varepsilon)
\] (14)

or

\[
\min_{\varepsilon \in \Lambda} \sigma_{total}(\varepsilon)
\] (15)

where the admissible set \(\Lambda\) for electrical permittivity \(\varepsilon\) satisfies the following two conditions:
• **Layered medium:** $\varepsilon(y)$ in $\Lambda$ is constant for each layer, either lossy or lossless. In other words, $\varepsilon(y) = \varepsilon_j$ at the $j$th layer, where $\varepsilon_j = \varepsilon_j' + i\varepsilon_j''$, $j = 1, 2, \cdots, L$, $L$ is the number of layers of the coating material.

• **Boundedness:** We assume there exist $\varepsilon_{\text{min}} = \varepsilon_{\text{min}}' + i\varepsilon_{\text{min}}''$, $\varepsilon_{\text{max}} = \varepsilon_{\text{max}}' + i\varepsilon_{\text{max}}''$ such that $\varepsilon_{\text{min}}' < \varepsilon_j' < \varepsilon_{\text{max}}'$ and $\varepsilon_{\text{min}}'' < \varepsilon_j'' < \varepsilon_{\text{max}}''$ for $j = 1, 2, \cdots, L$. Namely, the permittivity $\varepsilon(y)$ is bounded between $\varepsilon_{\text{min}}$ and $\varepsilon_{\text{max}}$.

Due to the finite number of layers, one can easily prove the admissible set $\Lambda$ is compact in $L^\infty(\Omega)$. Since the total field $u$ depends continuously on $\varepsilon \in L^\infty(\Omega)$, by the standard argument, it follows:

**Theorem 2.1.** The minimization problem (14) or (15) admits at least one solution in $\Lambda$.

**Remark 1.** It is possible to extend Thm. 2.1 to non-layered medium, in which case $\Lambda$ is only weak* compact in $L^\infty(\Omega)$. This, however, requires $u$ to be bounded in $H^1_g(\Omega)$ independent of the variation of $\varepsilon$ and the proof is far from trivial.

We intend to use a gradient based optimization algorithm for the RCS reduction. It is therefore very important to evaluate the gradient of the objective function accurately and efficiently, which will be shown in the next section.

3. **Gradient by the adjoint state method.** To determine the descent direction of the backscatter RCS (12) with respect to the permittivity, we need to evaluate the gradient along with the PDE constraint (9). It can be approximated by the numerical derivative, namely, finite difference. However, given the sensitive dependence of RCS on the permittivity and the amount of computational cost to evaluate (9), it is generally undesirable to use such a simple scheme. Our approach here is to derive the gradient for TE polarization based on the adjoint state method. This method is well-known for its effectiveness in evaluating the gradient when the state variable comes from a solution of a forward problem. Lots of nonlinear optimization problems have been studied based on this method; for example, the design of antireflective structure in [13]. Here we provide a formal derivation for the gradient based on the adjoint state method with proof omitted.

For convenience, let us first define the far field coefficient:

$$P_\theta(\varepsilon) = \frac{1}{2} \int_{\Gamma} \frac{\partial u}{\partial n} e^{-ik_0 x \cos \theta} dx,$$

Comparing with (12), we have

$$\sigma(\varepsilon) = \frac{4}{k_0} |P_\theta(\varepsilon)|^2$$

Let $\delta \varepsilon$ be a small perturbation to the permittivity $\varepsilon$. We denote the linearization of $\sigma(\varepsilon)$ with respect to $\delta \varepsilon$ by $\delta \sigma$, which is:

$$\delta \sigma = 2 Re \left( \frac{4}{k_0} \delta P \overline{P_\theta(\varepsilon)} \right)$$

(17)

where $\delta P$ denotes the linearization of $P_\theta(\varepsilon)$ and $\overline{P_\theta}$ is the complex conjugate of $P_\theta$. From (16), it is easy to see that

$$\delta P = \frac{1}{2} \int_{\Gamma} \frac{\partial \delta u}{\partial n} e^{-ik_0 x \cos \theta} dx$$

(18)
where \( \delta u \) solves the linearized problem:

\[
\begin{cases}
\nabla \cdot \left( \frac{1}{k^2} \nabla \delta u \right) + \delta u = \nabla \cdot \left( \frac{\omega^2 \mu_0 \delta \varepsilon}{k^4} \nabla u \right) \quad \text{in } \Omega \\
\frac{\partial \delta u}{\partial n} = 0 \quad \text{on } S \\
\delta u = T \left( \frac{\partial \delta u}{\partial n} \right) \quad \text{on } \Gamma
\end{cases}
\]

(19)

Let \( u^* \) solve the adjoint state equation:

\[
\begin{cases}
\nabla \cdot \left( \frac{1}{k^2} \nabla u^* \right) + u^* = 0 \quad \text{in } \Omega \\
\frac{\partial u^*}{\partial n} = 0 \quad \text{on } S \\
u^* = T^* \left( \frac{\partial u^*}{\partial n} \right) + \frac{1}{2} k^2 e^{ik_0 x \cos \theta} P_\theta (\varepsilon) \quad \text{on } \Gamma
\end{cases}
\]

(20)

where \( k^2 \) is the complex conjugate of \( k^2 \) and \( T^* (\cdot) \) is the adjoint operator of \( T (\cdot) \), given by

\[
T^* \left( \frac{\partial u^*}{\partial n} \right) = -\frac{1}{2 \pi i} \int_\mathbb{R} \frac{1}{\sqrt{k_0^2 - \xi^2}} \frac{\partial u^*}{\partial n} e^{i \xi \xi} d\xi
\]

Given equations (19) and (20), Green’s theorem implies that

\[
\int_\Omega u^* \nabla \cdot \left( \frac{\omega^2 \mu_0 \delta \varepsilon}{k^4} \nabla u \right) dx = \int_{\partial \Omega} \frac{1}{k^2} u^* \frac{\partial u}{\partial n} dx - \int_{\partial \Omega} \frac{1}{k^2} \frac{\partial u^*}{\partial n} \delta u dx
\]

(21)

Applying the integration by parts yields:

\[
\int_\Omega u^* \nabla \cdot \left( \frac{\omega^2 \mu_0 \delta \varepsilon}{k^4} \nabla u \right) dx = -\int_\Omega \nabla u^* \cdot \left( \frac{\omega^2 \mu_0 \delta \varepsilon}{k^4} \nabla u \right) dx + \int_{\partial \Omega} \frac{1}{k^2} \frac{\partial u^*}{\partial n} \delta u dx
\]

(22)

Recalling the homogeneous Neumann boundary condition, the second term on the right hand side of (22) is zero since the permittivity is constant near the aperture \( \Gamma \). Hence, combining (21) and (22) together with the property of adjoint operator, we obtain:

\[
-\int_\Omega \nabla u^* \cdot \left( \frac{\omega^2 \mu_0 \delta \varepsilon}{k^4} \nabla u \right) dx = \frac{1}{2} T^* (\varepsilon) \int_{\Gamma} \frac{\partial u}{\partial n} e^{-ik_0 x \cos \theta} dx
\]

(23)

Comparing with (17), we then have

\[
\delta \sigma = -2 Re \left( \frac{4}{k_0} \int_\Omega \nabla u^* \cdot \left( \frac{\omega^2 \mu_0 \delta \varepsilon}{k^4} \nabla u \right) dx \right)
\]

(24)

Up to a constant multiple, the gradient of the cost function \( \sigma (\varepsilon) \) is the function \( g (\varepsilon) \) such that

\[
\delta \sigma (\varepsilon) = Re \int_{-b}^{0} \delta \varepsilon g (\varepsilon) dy
\]

(25)

Comparing (24) with (25), we arrive at the formula for the gradient:

\[
g (\varepsilon) = -\frac{8}{k_0} \omega^2 \mu_0 \int_0^a \frac{1}{k^4} \left( \nabla u^* \right) \cdot \left( \nabla u \right) dx
\]

(26)

In other words, to find the gradient at a particular point \( \varepsilon \in \Lambda \), we need to solve the forward scattering problem (9) as well as the adjoint state problem (20).
Remark 2. The gradient given in (26) is in fact $L^2$ gradient, which is different from the usual vector form. In the numerical simulation, we need the finite dimensional counterpart of the gradient, which points to the maximal increase of RCS in the admissible set. Since the change $\delta \varepsilon$ is a constant for each layer, the finite discretization for (25) implies the gradient can be found through the following vector:

$$
(\int_{-b}^{-b+h} g(\varepsilon)dy, \int_{-b}^{-b+2h} g(\varepsilon)dy, \ldots, \int_{-b}^{-b+(L-1)h} g(\varepsilon)dy, )
$$

(27)

where $h$ is the thickness of each layer and $L$ is the number of layers. In practice, we can also split each element in (27) into real and imaginary part, which gives the corresponding change for the real and imaginary part of $\delta \varepsilon$, respectively.

For the cost function (13), since it is a linear combination of (12), the gradient can be easily deduced from (26). It can be seen that the adjoint problem shares the same structure as the original scattering problem, hence the same solver may be used for the adjoint problem. This is a huge advantage given the level of difficulties for solving the scattering problem from a cavity.

4. Optimization method: Sequential Quadratic Programming. Sequential Quadratic Programming (SQP) is a gradient based, iterative optimization method. Each iteration is composed by solving a quadratic subproblem. Each subproblem is based on the quadratic approximation of the Lagrangian function. In general, the Hessian matrix used in the quadratic subproblem is not obtained analytically, but approximated by some quasi-Newton methods. SQP provides a superlinear convergence near the region of the optimal point under some appropriate assumptions. Here we give a brief description of SQP to the RCS reduction problem. Detailed implementation can be found in [24].

Define the Lagrangian function:

$$
L(\varepsilon, \lambda) = \sigma(\varepsilon) + \sum_{j=1}^{4L} \lambda_j h_j(\varepsilon),
$$

(28)

where $\lambda_j$ is the Lagrangian multiplier, and $h_j(\varepsilon)$ gives the constraint on the permittivity:

$$
\begin{align*}
  h_{4(j-1)+1} &= -Re(\varepsilon_j) + \varepsilon'_\text{min} \\
  h_{4(j-1)+2} &= Re(\varepsilon_j) - \varepsilon'_\text{max} \\
  h_{4(j-1)+3} &= -Im(\varepsilon_j) + \varepsilon''_\text{min} \\
  h_{4(j-1)+4} &= Im(\varepsilon_j) - \varepsilon''_\text{max}
\end{align*}
$$

(29)

Here we consider the real and imaginary part of the permittivity separately. The boundedness of $\varepsilon$ is simply expressed as

$$
h_j(\varepsilon) \leq 0
$$

(30)

Following the theory of SQP, the $m$-th subproblem during the iteration is given by:

$$
\min_{d \in \mathbb{C}^L} \sigma(\varepsilon_m) + g(\varepsilon_m)^T(d) + \frac{1}{2} d^T H_m d
$$

(31)
with the linear constraint (29) satisfied by \( \varepsilon_m + d \), where \( g(\varepsilon_m) \) is the gradient of \( \sigma(\varepsilon) \) at \( \varepsilon_m \), \( H_m \) is the Hessian of the Lagrangian function (28), which in this paper is approximated by Broyden-Fletcher-Goldfarb-Shanno (BFGS) method.

Solution of the subproblem (31) provides an update for \( \varepsilon \) along the direction \( d \)

\[
\varepsilon_{m+1} = \varepsilon_m + \alpha d
\]  

(32)

The step length \( \alpha \in (0, 1) \) can be determined by an appropriately defined merit function. We use the following \( L^1 \) merit function

\[
\varphi(\varepsilon) = \sigma(\varepsilon) + \sum_{j=1}^{4L} \mu_j \max(0, h_j(\varepsilon))
\]  

(33)

for sufficiently large \( \mu_j > 0 \), which gives penalty for solutions outside the admissible set.

We are now ready to state the minimization algorithm:

**Data:** Initialize permittivity \( \varepsilon_0 \), tolerance \( \epsilon \), penalty coefficients \( \mu_j \), for \( j = 1, \ldots, 4L \).

Let \( m = 0 \), \( \alpha = 1 \);

while \( \alpha > \epsilon \) do

Evaluate \( g(\varepsilon_m), H_m, h(\varepsilon_m) \);
Solve (31) to obtain \( d \);
Set \( \varepsilon_{m+1} \leftarrow \varepsilon_m + \alpha d \);
while \( \varphi(\varepsilon_{m+1}) > \varphi(\varepsilon_m) \) do

\[ \alpha \leftarrow \alpha / 2 \]
Update \( \varepsilon_{m+1} \leftarrow \varepsilon_m + \alpha d \);
end
end

**Remark 3.** Since the problem (14) we consider is generally not convex with respect to \( \varepsilon \), the solution from the algorithm above may not converge to the global one. In practice, we either expect a good initial guess is provided, or try different initial values in the parameter space and choose the most optimal one. More complicated strategy is possible, but we will not pursue it here.

Throughout the algorithm, it is required to provide the objective function, constraint, gradient and a direct solver for the forward scattering problem (9). We already discussed how to find the gradient in Section 3. In the next section, we present an accurate and efficient direct solver for the optimization algorithm.

5. **Direct solver for the forward scattering problem.** Numerical solution for the electromagnetic scattering from a cavity has long been known as a difficult problem [21], especially when the size of the cavity is large or the wavenumber is high. Here we extend the finite difference method developed in [8] to be the direct solver of the optimization problem. The idea to solve the resulting linear system in [8] is to use the discrete cosine transform in the horizontal direction and a Gaussian elimination process in the vertical direction. The resulting system becomes an \( M \times M \) (\( M \) is number of mesh points at the opening of the cavity) linear system, which is much smaller than the original one. However, the original algorithm in [8] can not directly apply to the design problem here since it requires the permittivity \( \varepsilon \) of the medium inside the cavity to be differentiable in the vertical direction for TE polarization, which is not satisfied for layered medium. Therefore, we modify
the algorithm to make it applicable for our case. The idea is simple: we apply finite difference in each layer and connect each layer by the continuity condition, while still keep the resulting system suitable for the discrete cosine transform in the horizontal direction.

Assume \( \Omega \) is composed by \( L \) layers with each layer denoted by \( \Omega_l \), \( l = 1, \ldots, L \) and the associated permittivity is given by \( \varepsilon_l \). Similar to [8], the interior and the aperture of the cavity is uniformly partitioned by the mesh points \( \{x_m, y_n\}_{m=1}^{M+1},_{n=1}^{N+1} \). We further assume there exist mesh points on each interface of two layers with \( n = n_1, n_2, \ldots, n_{L-1} \). The algorithm will not lose the generality by this assumption since the extension to nonuniform mesh in \( y \) direction is straightforward. Define \( u_{m,n} \) as the corresponding numerical solution at \( (x_m, y_n) \). Since the material is homogeneous in each layer, the second order finite difference scheme in the \( l \)-th layer is simply given by the following:

\[
\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{h_x^2} + \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{h_y^2} + \frac{\omega^2 \mu_0 \varepsilon_l u_{m,n} = 0}{h_y^2},
\]

where \( (x_m, y_n) \in \Omega_l \) (34)

Equations at different layers are coupled together via the continuity condition:

\[
u^+ = u^-,
\]

\[
\frac{1}{\varepsilon_l^+} \frac{\partial u^+}{\partial y} = \frac{1}{\varepsilon_l^-} \frac{\partial u^-}{\partial y},
\]

where ‘+’ and ‘−’ denote the upper and lower side of an interface, respectively.

Condition (35) implies it is unnecessary to distinguish the value on the two sides of an interface. Condition (36) can be approximated by the finite difference with points on the same vertical line. Here we use the second order approximation in order to be consistent with (34). For \( n = n_1, n_2, \ldots, n_{L-1} \), the approximation is given by:

\[
\frac{1}{\varepsilon_{l+1}} \frac{1}{2} \frac{1}{2} \frac{1}{2} u_{m,n+2} - 2u_{m,n+1} + 3/2 u_{m,n} = \frac{1}{\varepsilon_{l}} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} u_{m,n+2} - 2u_{m,n+1} + 3/2 u_{m,n} - 2u_{m,n-2} + 2u_{m,n-1} - 3/2 u_{m,n}
\]

For simplicity, the first order approximation is applied to the homogenous Neumann boundary condition at \( \partial \Omega \). It is also used to discretize the normal derivative and the improper integral at the aperture \( \Gamma \), which results in the following discretization[17]:

\[
u_{m,N+1} = \sum_{n=1}^{N} G_{m,n} \left( \frac{u_{m,N+1} - u_{m,N}}{h_y} \right) + g_m, \quad m = 1, 2, \ldots, M
\]

where

\[
G_{m,n} = \begin{cases} 
\frac{1}{2} \left[ 1 + \frac{2i}{\pi} \ln(0.1638k_0h_x) \right] & \text{if } n = m \\
\frac{1}{2} H_0^{(2)} \left( k_0 |x_n - x_m| \right) & \text{if } n \neq m
\end{cases}
\]

\[
g_m = 2e^{i\alpha n}.
\]

For compactness, the scheme can be rewritten in the matrix form:

\[
(A_x \otimes I_N + I_M \otimes A_y + I_M \otimes D^*)u + (I_M \otimes a_{N+1})u_{N+1} = 0
\]

\[
Gu_N + (h_y I_M - G) u_{N+1} = h_y g
\]
where $\otimes$ denotes the tensor product (Kronecker product), $I_M$ is the $M \times M$ identity matrix, $I_N^*$ is similar to the $N \times N$ identity matrix but the entry at $(n_l, n_l)$ is zero for $l = 1, 2, \cdots, L - 1$, and $G = (G_{m,n})_{m=1,n=1}^M$.

For the other matrices, their forms are:

$$A_x = \frac{1}{h_x^2} \begin{pmatrix} -1 & 1 & & & & \vdots & & & \end{pmatrix}_{M \times M} a_{N+1} = \frac{1}{h_y^2} \begin{pmatrix} 0 & & & & & \vdots & & & \\ \vdots & & & & & & & & \end{pmatrix}_{(N+1) \times 1}$$

$$A_y^* = \frac{1}{h_y^2} \begin{pmatrix} -1 & 1 & & & & \vdots & & & \frac{1}{2} \varepsilon_{l+1} \varepsilon_l & -2 \varepsilon_{l+1} \varepsilon_l & \frac{3}{2} \left(1 + \frac{\varepsilon_{l+1}}{\varepsilon_l}\right) & -2 & \frac{3}{2} & -2 & 1 & \vdots & \vdots & \vdots \frac{1}{2} \varepsilon_{l+1} \varepsilon_l & -2 \varepsilon_{l+1} \varepsilon_l & \frac{3}{2} \left(1 + \frac{\varepsilon_{l+1}}{\varepsilon_l}\right) & -2 & \frac{3}{2} & -2 & 1 & \vdots & \vdots & \vdots \frac{1}{2} \varepsilon_{l+1} \varepsilon_l & -2 \varepsilon_{l+1} \varepsilon_l & \frac{3}{2} \left(1 + \frac{\varepsilon_{l+1}}{\varepsilon_l}\right) & -2 & \frac{3}{2} & -2 & 1 & \vdots & \vdots & \vdots \end{pmatrix}_{N \times N}$$

and

$$D^* = \text{diag}(\varepsilon_1, \cdots, \varepsilon_1, 0, \varepsilon_2, \cdots, \varepsilon_{L-1}, 0, \varepsilon_L, \cdots, \varepsilon_L),$$

$$u = (u_{1,1}, \cdots, u_{1,N}, u_{2,1}, \cdots, u_{2,N}, \cdots, u_{M,1}, \cdots, u_{M,N})^T,$$

$$u_n = (u_{1,n}, \cdots, u_{M,n})^T$$

where $D^*$ is a diagonal matrix with zero appearing at the $n_l$th row, $l = 1, 2, \cdots, L$.

We then apply the discrete cosine transform to $A_x$ and the rest of the algorithm is completely analogy to the method introduced in [8], which we will omit and refer the readers to [8] for details. We also omit the proof for the unique solvability of the resulting linear system (41) for conciseness. As the algorithm highly utilizes the special structure of the cavity problem, it is extremely fast and efficient, which satisfies the requirement for a computation engine of the optimization method.

6. Numerical experiments. In this section, we apply our algorithm to the RAM design through three numerical examples. Throughout, we assume the dimension of the cavity is $1m$ in width and $0.3m$ in depth. In other words, $\Omega = [0, 1] \times [-0.3, 0]$. We partition $\Omega$ uniformly by a $512 \times 512$ mesh. The thickness of the coating material for each layer is $2.3mm$, with four layers in total. Assume the relative permittivity is a continuous variable between $1 + 0i$ and $100 + 100i$. Note that these settings are only used for illustration purpose. Our algorithm applies to any finite number of layers and any finite bounds of the permittivity. All the numerical experiments are carried out on a laptop with 2.1Ghz Intel dual core and 4GB memory. For physical
interest, the RCS is expressed in terms of decibel(dB), which is

\[
\text{RCS}: = 10 \log_{10} \sigma \text{ dB}.
\]

Initially, we assume the layered RAM has a uniform permittivity, namely, \(5 + 5i\).

As we mentioned before, the resulted permittivity of the material after the optimization may highly depend on the initial value. We certainly have tried other initial values, but no significant improvement was observed in terms of RCS reduction. Therefore, all the examples shown here are based on the same initial value.

**Example 1:** Consider the scattering from a cavity by the incidence of a plane wave with wavenumber \(k_0 = 16\pi\). Figure 2(a) gives the result for the optimization at \(\theta = \frac{3\pi}{5}\). RCS is reduced by more than 100dB at \(\theta = \frac{3\pi}{5}\). The reduction is not only confined around \(\theta = \frac{3\pi}{5}\). It also extends to the other angular sector with an average reduction around 10dB, which can be considered as a large improvement. Similar effect has been shown in Figure 2(b), which gives the RCS when the optimization is conducted at \(\theta = \frac{5\pi}{6}\). The deep well in the graph shows a large amount of reduction at \(\theta = \frac{5\pi}{6}\). Meanwhile, the RCS at the other angular sector also gets reduced. Figure 2(c) shows the RCS reduction when the optimization is conducted by the combination of \(\theta = \frac{3\pi}{5}\) and \(\theta = \frac{5\pi}{6}\). The weights for these two angles are set to be equal. Compared to the optimization at a single angle, optimization with multiple angles avoids the sharp reduction at a particular angle and thus provides a smoother reduction for all the angles. The resulted relative permittivities are listed in Table 1 with the corresponding number of iterations and CPU time. Generally, the computational time for optimization with two angles are much longer than the time spent for a single angle. These observations are consistent to the optimization result in TM polarization[7].

**Example 2:** Consider the incidence of a plane wave with wavenumber \(k_0 = 32\pi\) on the cavity. Similar to Example 1, the optimization are performed at \(\frac{3\pi}{5}, \frac{5\pi}{6}\) and both. Figure 3(a) gives the result for the optimization at \(\theta = \frac{3\pi}{5}\). RCS are reduced by more than 100dB at \(\theta = \frac{3\pi}{5}\). At the other angular sectors, RCS also gets reduced, although the performance is not as good as \(\theta = \frac{3\pi}{5}\). Figure 3(b) shows the result for RCS optimization at \(\theta = \frac{5\pi}{6}\). The deep well in the graph shows a large amount of reduction at \(\theta = \frac{5\pi}{6}\). Meanwhile, the reduction is also obtained at the other angles. Figure 3(c) gives the optimized RCS when the optimization is conducted by the combination of \(\theta = \frac{3\pi}{5}\) and \(\theta = \frac{5\pi}{6}\). The sharp reduction at a particular angle disappears but the overall reduction gets improved. Table 1 contains all the resulted relative permittivities with the corresponding number of iterations and CPU time. It again shows more time is needed for the optimization over the combination of multiple angles. As shown in [7], the first few steps during the iteration achieves most of the reduction for multiple angles. The computational time therefore can be significantly reduced by terminating the algorithm if only minor improvement appears.

**Example 3:** In the last example, we test our algorithm by showing a RCS reduction over a range of frequencies. In particular, we consider the reduction between 16\(\pi\) and 32\(\pi\), which implies the frequency ranges from 2.4Ghz to 4.8Ghz. Take the objective function as a combination of RCS at \(k = 16\pi\) and \(k = 32\pi\). The formula is given in (13) with equal weights for the two wavenumber. Results are shown in Figure 4 for both \(\theta = \frac{\pi}{2}\) and \(\theta = \frac{3\pi}{5}\) and the corresponding CPU time are given in Table 1. From Figure 4(a), we see some reduction at the two end. However, the overall effect is merely some kind of horizontal shift of RCS. Generally it implies a
Figure 2. RCS reduction for $k_0 = 16\pi$. (a) Optimized at $\theta = \frac{3\pi}{5}$. (b) Optimized at $\theta = \frac{5\pi}{6}$. (c) Optimized with the combination of $\theta = \frac{3\pi}{5}$ and $\theta = \frac{5\pi}{6}$.

reduction at one frequency accompanies an enhancement at another frequency for normal incidence. The situation is changed for oblique incidence, as shown in 4(b), which gives an overall reduction for all the frequencies. As mentioned in [7], the difference between these two can be explained by ray tracing. For normal incidence, most of the rays are directly reflected back to the same direction. While for oblique incidence, rays are bouncing back and forth inside the cavity, so more energy can be absorbed by the RAM as compared to normal incidence. In addition, most of the rays shooting out of the cavity are not in the same direction as they come from. Therefore, more reduction is observed for oblique incidence.

7. Conclusion. In this paper, the design of a multilayered RAM for reducing RCS of a cavity is formulated as a minimization problem. The descent direction for the cost function is evaluated through the adjoint state method. Subsequently, SQP is integrated with the gradient to obtain the optimal absorbing materials for the cavity. The algorithm is implemented with a fast and accurate direct solver for the scattering problem. Numerical results show that RCS is reduced significantly with
Figure 3. RCS reduction for $k_0 = 32\pi$. (a) Optimized at $\theta = \frac{3\pi}{5}$. (b) Optimized at $\theta = \frac{5\pi}{6}$. (c) Optimized with the combination of $\theta = \frac{3\pi}{5}$ and $\theta = \frac{5\pi}{6}$.

Figure 4. RCS reduction for $k$ between $16\pi$ and $32\pi$. (a) Optimized at $\theta = \frac{\pi}{2}$. (b) Optimized at $\theta = \frac{3\pi}{5}$. 
RCS REDUCTION FOR A CAVITY IN TE POLARIZATION

Example 1

\[ k = \frac{16\pi}{4}, \quad \frac{16\pi}{3}, \quad \frac{16\pi}{5}, \quad \frac{16\pi}{7} \]

\[ \theta = \frac{\pi}{3}, \quad \frac{\pi}{5}, \quad \frac{\pi}{6}, \quad \frac{\pi}{7} \]

1st (Re) 4.669 5.374 1.000 4.809 5.004 1.000 1.000 93.098
1st (Im) 6.737 5.066 3.293 4.925 4.936 0.000 0.009 62.367

2nd (Re) 6.786 7.005 1.000 3.659 4.793 54.316
2nd (Im) 6.341 4.837 2.662 4.370 4.775 25.262

3rd (Re) 11.107 10.176 29.271 2.892 4.684
3rd (Im) 6.767 4.931 0.000 2.732 4.384

4th (Re) 16.573 13.985
4th (Im) 9.981 6.613

Iter. 15 16 100 10 11 129 24 81
CPU 48.9s 52.7s 1113s 50.5s 49.1s 2404s 33.8s 1420s

Table 1. Relative permittivities after the optimization, number of iterations and CPU time for all the three examples.

a thin RAM coated at the bottom of the cavity. It is also observed that different weights in the cost function generally lead to different RAM design and RCS reduction. Future work includes considering RCS reduction for three dimensional cavities.

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